Prerequesites: Levels 8.1 and 8.2.

We will aim to construct a function on $[0, 1]$ with the following properties:

- It is differentiable everywhere

- Its derivative is bounded

- Its derivative is not Riemann integrable

To do this we will invoke a theorem from level 8.2 which says that a bounded function is Riemann integrable if and only if it is discontinuous on a set of measure 0. Therefore, we aim to make a function which is discontinuous on a set which is not measure 0.

Note that by "measure 0", we mean "can be covered by intervals of total length $< \varepsilon$" and we have not yet shown in levels 1-8 that this is equivalent to "measure 0" by the definition of length we briefly gave in level 6.2. Therefore, we will stick with the open intervals definition for the purposes of this document.

First, it is not even easy to come up with a derivative with a single point of discontinuity. If you try to make a jump discontinuity, what will happen is your function just won't even be differentiable at that point. For a function to genuinely be differentiable yet its derivative is not continuous at said point, is not trivial. But my example of how we can make a single point discontinuity will be useful for better understanding the problem.

Our example of a derivative with one point of discontinuity will be the function $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ and $f(0) = 0$. Figure 1 shows the graph of what this looks like:

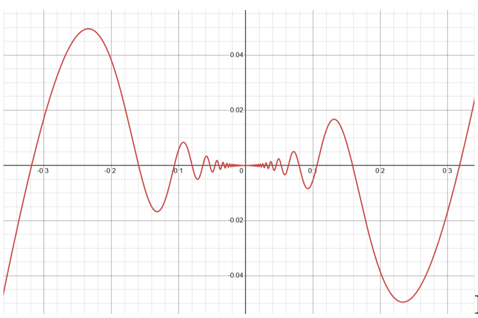

Figure 1

This is clearly differentiable away from 0 with derivative $2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ by the product and chain rules.

At 0, we note that as $h \to 0$,

$$\left|\frac{f(h) - f(0)}{h}\right| = \left|\frac{f(h)}{h}\right| = \left|h \sin\left(\frac{1}{h}\right)\right| \leq |h| \to 0$$

Thus we have $f'(0) = 0$. Note though that near 0, $2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ will behave wildly, as what happens is $2x \sin\left(\frac{1}{x}\right)$ approaches 0 but $\cos\left(\frac{1}{x}\right)$ bounces between -1 and 1 faster and faster as $\frac{1}{x} \to \infty$. Figure 2 shows a graph of the derivative and it is easy to see that there is no way that the derivative will be continuous at 0, despite existing everywhere. The trick was essentially to make the derivative oscillate like this since a simple jump discontinuity will make the function not be differentiable in the first place.

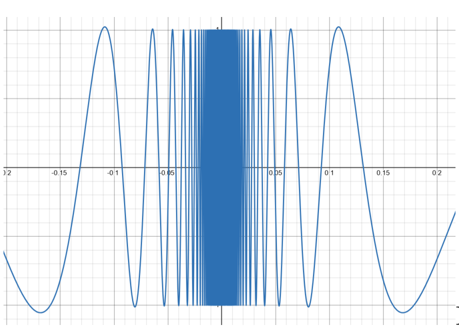

Figure 2

Note that if we drop the "bounded" condition, the construction becomes much easier: A function like $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ will be differentiable for the same reason but if we compute its derivative we see that it will be unbounded, and unbounded functions are not Riemann integrable as we define Riemann integrable functions to have to be bounded, however the improper integral of $f'(x)$ in this case still exists.

For convenience, we will use $f(x) = \begin{cases} x^2 \sin\left(\frac{\pi}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ as a basis for our construction. This satisfies $f(1) = 0$. We will take the piece of f from 0 to 1 and rotate it by 180 degrees about the point $(1, 0)$ to get a function on $[0, 2]$ that is differentiable everywhere but its derivative oscillates wildly near 0 and 2. Figure 3 shows this function, which we will use for our construction.

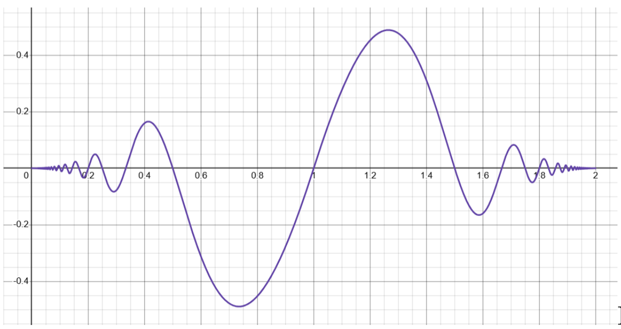

Figure 3

This is differentiable at the point of rotation with the original derivative, and this is clear. Note that $f\left(\frac{1}{k}\right)$ for any positive integer k is also 0 so we could reflect about these points as well and still have a valid similar thing on $[0, \frac{2}{k}]$. We will use this.

Before we do the construction I will construct the set that we aim to make the derivative discontinuous on and prove that this set does not have measure 0.

Figure 4 shows a construction of the cantor set. The way this works is we start with $[0, 1]$ and then remove the middle third so we end up with 2 closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. We will remove the middle third from each of these intervals to get 4 closed intervals, and continue. We will then take the intersection of all these closed intervals and we will get the cantor set. The cantor set will contain the endpoints, such as $\frac{1}{3}$ and $\frac{2}{9}$, as well as some other points like $\frac{1}{4}$ that did not get removed.

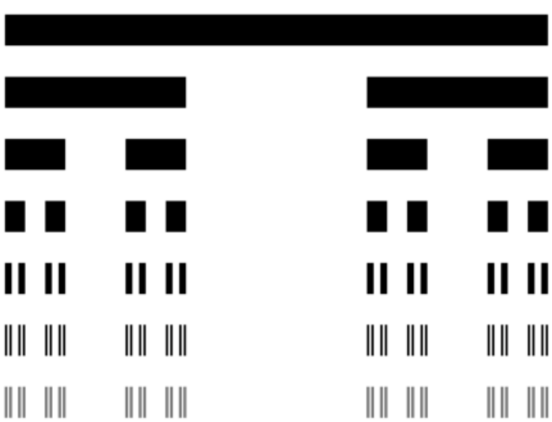

Figure 4

The issue, though, is that this set has measure 0. However, there is a nice modification we can make.

We can construct the "fat cantor set" by removing intervals of size $\frac{1}{4^n}$ from the middle of each closed interval on the n'th step, as opposed to $\frac{1}{3^n}$. See Figure 5 for a visual of the fat cantor set.


Figure 5

We will make a function which has a derivative which is not continuous anywhere on the fact cantor set, but yet the derivative is bounded. By the next proposition, it will follow that the derivative of the function we will construct, if we successfully construct it, will indeed be non-integrable.

Proposition. The fat cantor set does not have measure 0.

Proof. We will prove that there is no set of open intervals that cover the fat cantor set such that their total length is $< \frac{1}{2}$. Since it is not possible to have uncountably many open intervals with each non-zero length and finite total length (I've proven this somewhere in level 6 or 7), we can assume for a contradiction that there are countably many open intervals with this property.

Take a fictitious such covering of the fat cantor set with open intervals. Now add to this covering all the open intervals we remove from the construction of the fat cantor set to get a cover of the closed set $[0,1]$ by open intervals. This will include the interval $\left(\frac{3}{8}, \frac{5}{8}\right)$ as well as 2 intervals of length $\frac{1}{16}$, 4 of length $\frac{1}{64}$, and at the n'th step it will include 2^{n-1} intervals each of length $\frac{1}{4^n}$. The total length of the intervals we will add to get a cover for $[0,1]$ will therefore be $\sum_{r=1}^{\infty} \frac{2^{n-1}}{4^n} = \frac{1}{2} \sum_{r=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}$.

Therefore, by our contradition assumption, we end up with a covering of all of $[0,1]$ by open intervals of total length $\frac{1}{2} + \left(< \frac{1}{2}\right)$ which is < 1.

Now you may think this is already a contradiction so we are done, and that would be true, but it is not obvious since this fails if we consider the set of rational numbers between 0 and 1 in which you may come to the conclusion that it cannot have measure 0 even though I could prove to you that in fact it does, so we have to be careful.

So we have this weird covering of $[0,1]$ with total length < 1. Since $[0,1]$ is a closed and bounded subset of $\mathbb{R}$, I will have proven in either level 7 or 8 somewhere that it has the property that every open cover of $[0,1]$ has a finite subcover. Thus, take a finite subcover of the cover above. This will be a FINITE cover of $[0,1]$ with total length < 1, and THIS is sufficiently obvious that it is not possible. Therefore we conclude.

$\square$

Now we begin the construction.

We essentially make a modification of figure 3 where instead of using 1 as our point to rotate about we use $\frac{1}{8}$. We then place this modification, which is of total length $\frac{1}{4}$, into the first interval we remove which is $\left(\frac{3}{8}, \frac{5}{8}\right)$. We then do another modification where now the point we rotate about is $\frac{1}{32}$. This gives us something analagous to figure 3 which we will place into the 2 intervals we removed at the second step of the fat cantor set construction. We will do the same thing; In general we will modify figure 3 such that the point we rotate about is $\frac{1}{2*4^n}$ and place it in each of the 2^n intervals we removed at the n'th step in the fat cantor set construction.

The resulting function after we do this bizarre construction is Volterra's function. Figure 6 shows a sketch of this, although it is difficult to see the inserted things past the second step, since they get very small.

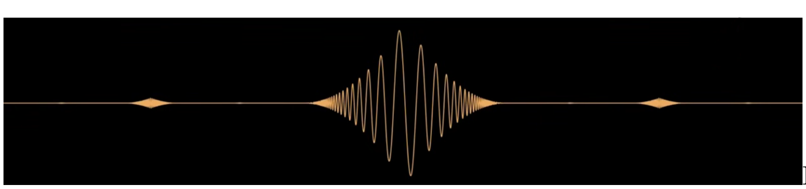
Figure 6

If we zoom in we will see more smaller copies of the middle bit.

The function is clearly differentiable inside these open intervals, ie not in the cantor set. However, what is also clear is that near the 2 boundary points of each of these open intervals, the derivative will oscillate wildly.

Theorem. At any point in the fat cantor set, Volterra's function has derivative 0, and yet the derivative is discontinuous.

Proof. To see that it has derivative 0 at x for x in the fat cantor set note that $\frac{f(x+h)-f(x)}{h} = \frac{f(x+h)}{h}$. If $x+h$ is in the cantor set this is 0, otherwise it is in one of the open intervals. If $h > 0$ we will let k be such that $x+k$ is the left boundary of this open interval, and if $h < 0$ we will let k be such that $x+k$ is at the right boundary of this open interval. Clearly, k lies between 0 and h so we have

$$\left|\frac{f(x+h)}{h}\right| \leq \left|\frac{f((x+k)+h-k)}{h-k}\right|$$

But as $h \to 0$, $h-k$ also goes to 0. Crucially, we know that by our construction

$$\left|\frac{f((x+k)+h-k)}{h-k}\right| \leq \frac{|h-k|^2}{|h-k|} \to 0$$

Thus the difference quotient goes to 0 so the derivative is 0. The last inequality is since any reflection of figure 3 will always be bounded above and below by the parabolas $\pm(z-(x+k))^2$ where z is the independent variable, and this is independent of which interval $x+h$ is in, hopefully this makes sense.

Now we just need to show that the derivative is discontinuous. For any x in the fat cantor set, note that it is clear that after the n'th step it is true that any point in the set from the n'th iteration of the construction is within a distance of $\frac{1}{2^n}$ units from a removed interval. Therefore, in the final function, in any interval about x, no matter how small, it will contain an area where the derivative oscillates wildly, so the derivative will not be continuous there.

□

This completes the construction and proof. If you're like me you'll think wow this is so cool and fun. If you think what's the point because we're only interested in well behaved stuff, then, well, I don't know what to tell you.