We will construct a function which is continuous everywhere but differentiable nowhere.

Prerequesites: Levels 1-4. Idea of uniform convergence from level 6, and the fact from the Analysis II cambridge course in level 9 that a uniform limit of continuous functions is continuous. Knowledge of what "mod 2" means from numbers and sets (level 7.1).

Theorem. The function $\sum_{n=0}^{\infty} \frac{\cos(13^n \pi x)}{2^n}$ is continuous everywhere

Proof. It is easy to see that if we replace ∞ in the sum then we will have a continuous function.

It will be easy to see that we have a continuous function if the convergence is uniform. We can establish this by making the tail small in a way that does not depend on x.

Note that

$$\left| \sum_{n=k+1}^{\infty} \frac{\cos\left(13^n \pi x\right)}{2^n} \right| \leq \sum_{n=k+1}^{\infty} \left| \frac{\cos\left(13^n \pi x\right)}{2^n} \right| \leq \sum_{n=k+1}^{\infty} \left| \frac{1}{2^n} \right| = \frac{1}{2^k} \to 0$$

□

We claim that this function is nowhere differentiable. We will call this function $W(x)$. This function is "the" weierstrass function, although it is a family of functions where we can change "13" and "2" around.

If you have not seen this before, try to imagine what a nowhere differentiable function might look like. It is certainly difficult, since any attempt you make at drawing a continuous line will give you a line that has a slope. After the proof I will show an image of the graph.

To do this, for any x we will find a sequence of increments h_m such that

$$\left| \frac{W\left(x + h_m\right) - W\left(x\right)}{h_m} \right| \to \infty \text{ as } m \to \infty$$

We will define

$$S_m\left(x\right) := \sum_{n=0}^{m-1} \frac{\cos\left(13^n \pi x\right)}{2^n}$$

$$R_m\left(x\right) := \sum_{n=m}^{\infty} \frac{\cos\left(13^n \pi x\right)}{2^n}$$

So we have $S_m\left(x\right) + R_m\left(x\right) = W\left(x\right)$ for all m, x.

Now we will start the proof.

Theorem. $W(x)$ is nowhere differentiable.

Proof. Note that we cannot just differentiate each term in the sum since as we showed in Level 4 that this does not always work for infinite sums.

For a fixed x and an integer $m > 0$ write $13^m x = \alpha_m + e_m$ where α_m is the closest integer to $13^m x$ so that $-\frac{1}{2} < e_m \leq \frac{1}{2}$

We will define

$$h_m := \frac{1 - e_m}{13^m}$$

This clearly $\to 0$ since the numerator is bounded but the denominator $\to \infty$. We will later use a bound that $\frac{1}{|h_m|} > \frac{2*13^m}{3}$ which follows from the fact that the numerator is upper bounded by $\frac{3}{2}$.

We know that $S_m(x)$ is differentiable with derivative

$$S_m'(x) = -\sum_{n=0}^{m-1} \pi \left(\frac{13}{2}\right)^n \sin(13^n \pi x)$$

We have

$$|S_m'(x)| \leq \sum_{n=0}^{m-1} \pi \left(\frac{13}{2}\right)^n \leq 2\frac{\left(\frac{13}{2}\right)^m}{11}\pi$$

Thus, this maximum absolute value of the derivative is the maximum "change in y over change in x". Specifically we have

$$\left|\frac{S_m(x+h_m) - S_m(x)}{h_m}\right| \leq 2\frac{\left(\frac{13}{2}\right)^m}{11}\pi$$

It is not clear why we need this but we will see later.

Now we note that if $n \geq m$ then

$$13^n(x+h_m) = 13^{n-m}13^m\left(x + \frac{1-e_m}{13^m}\right) = 13^{n-m}(13^m x + 1 - e_m)$$

$$= 13^{n-m}(\alpha_m + 1)$$

Since 13^{n-m} is odd, $13^n(x+h_m)$ is an integer equal to $(1+\alpha_m) \bmod 2$. This is very good because now we know that

$$\cos(13^n \pi(x+h_m)) = \cos(\pi(\alpha_m+1)) = -(-1)^{\alpha_m}$$

Now consider $R_m(x+h_m)$. This equals

$$\sum_{n=m}^{\infty} \frac{\cos(13^n\pi(x+h_m))}{2^n} = \sum_{n=m}^{\infty} \frac{-(-1)^{\alpha_m}}{2^n} = -(-1)^{\alpha_m} 2^{1-m}$$

Next we evaluate this tail at the original x. We have

$$R_m(x) = \frac{\cos(13^m\pi x)}{2^m} + \sum_{n=m+1}^{\infty} \frac{\cos(13^n\pi x)}{2^n}$$

$$= \frac{\cos(e_m\pi + \alpha_m\pi)}{2^m} + \sum_{n=m+1}^{\infty} \frac{\cos(13^n\pi x)}{2^n}$$

$$= \frac{\cos(e_m\pi)}{2^m}(-1)^{\alpha_m} + \sum_{n=m+1}^{\infty} \frac{\cos(13^n\pi x)}{2^n}$$

Now we have that

$$R_m(x+h_m) - R_m(x) = -(-1)^{\alpha_m}2^{1-m} - \frac{\cos(e_m\pi)}{2^m}(-1)^{\alpha_m} - \sum_{n=m+1}^{\infty} \frac{\cos(13^n\pi x)}{2^n}$$

$$= -(-1)^{\alpha_m}\left(2^{1-m} + \frac{\cos(e_m\pi)}{2^m} + (-1)^{\alpha_m}\sum_{n=m+1}^{\infty} \frac{\cos(13^n\pi x)}{2^n}\right)$$

Let A be the stuff inside the brackets. Then we know that

$$A \geq 2^{1-m} + \frac{\cos(e_m\pi)}{2^m} + \sum_{n=m+1}^{\infty} \frac{-1}{2^n} = 2^{-m} + \frac{\cos(e_m\pi)}{2^m} = 2^{-m}(1+\cos(e_m\pi))$$

Since e_m is between $-\frac{1}{2}$ and $\frac{1}{2}$ we know that $\cos(e_m\pi) \geq 0$ so $A \geq \frac{1}{2^m}$.

This means that

$$|R_m(x+h_m) - R_m(x)| \geq \frac{1}{2^m}$$

Thus

$$\left|\frac{R_m\left(x+h_m\right)-R_m\left(x\right)}{h_m}\right| \geq \frac{1}{2^m|h_m|} \geq \frac{2}{3}\left(\frac{13}{2}\right)^m$$

By our bound on $\frac{1}{|h_m|}$ from earlier.

We know that

$$\frac{R_m\left(x+h_m\right)-R_m\left(x\right)}{h_m} = \frac{W_m\left(x+h_m\right)-W_m\left(x\right)}{h_m} - \frac{S_m\left(x+h_m\right)-S_m\left(x\right)}{h_m}$$

So the triangle inequality gives

$$\left|\frac{R_m\left(x+h_m\right)-R_m\left(x\right)}{h_m}\right| \leq \left|\frac{W_m\left(x+h_m\right)-W_m\left(x\right)}{h_m}\right| + \left|\frac{S_m\left(x+h_m\right)-S_m\left(x\right)}{h_m}\right|$$

Thus

$$\left|\frac{W_m\left(x+h_m\right)-W_m\left(x\right)}{h_m}\right| \geq \left|\frac{R_m\left(x+h_m\right)-R_m\left(x\right)}{h_m}\right| - \left|\frac{S_m\left(x+h_m\right)-S_m\left(x\right)}{h_m}\right|$$

By our bounds from earlier we get

$$\left|\frac{W_m\left(x+h_m\right)-W_m\left(x\right)}{h_m}\right| \geq \frac{2}{3}\left(\frac{13}{2}\right)^m - 2\frac{\left(\frac{13}{2}\right)^m}{11}\pi$$

$$= \left(\frac{13}{2}\right)^m\left(\frac{2}{3}-\frac{2}{11}\pi\right)$$

Since $\pi < \frac{11}{3}$ this is not negative (There are many ways to prove this, including by considering the perimeter of a regular hexagon outside a circle being $2\sqrt{3}$ which is between π and $\frac{11}{3}$), we conclude that the difference quotient does indeed go to infinity for this sequence h_m, so W is continuous but nowhere differentiable.

$\square$

Now we show images.

Figure 1 shows the graph of W(x).

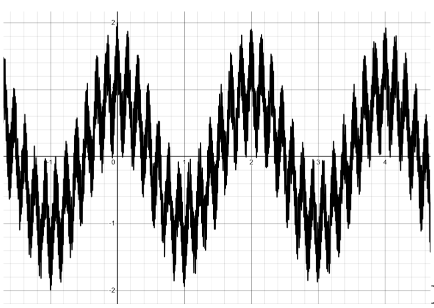
Figure 1

Figure 2 shows the graph zoomed in a lot.

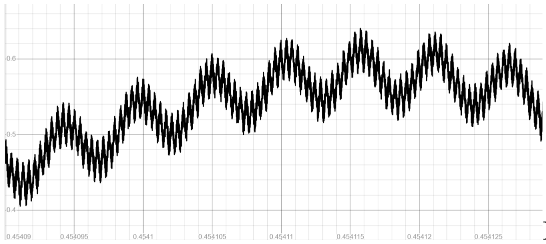
Figure 2

Interesting to see how what we proved is reflected in the pictures! The graph essentially is a fractal curve that looks the same at all magnifications.