Theorem: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Levels recommended for proof: 5

Proof:

Recall De Moivre's theorem, which says that $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$. Now we have that $\frac{(\cot(\theta)+i)^{2n+1} - (\cot(\theta)-i)^{2n+1}}{2i} = \frac{(\cos(\theta)+i\sin(\theta))^{2n+1} - (\cos(\theta)-i\sin(\theta))^{2n+1}}{2i(\sin^{2n+1}(\theta))} = \frac{\cos((2n+1)\theta)+i\sin((2n+1)\theta) - \cos((2n+1)\theta)+i\sin((2n+1)\theta)}{2i(\sin^{2n+1}(\theta))} = \frac{\sin((2n+1)\theta)}{\sin^{2n+1}(\theta)}$.

Now lets suppose that $\theta = \frac{m\pi}{2n+1}$ for some integer m that is between 1 and n inclusive. Then we know that $\sin\big((2n+1)\theta\big) = 0$ and therefore $\frac{(\cot(\theta)+i)^{2n+1} - (\cot(\theta)-i)^{2n+1}}{2i} = 0$. Now I will use the binomial theorem to expand $(\cot(\theta) + i)^{2n+1} - (\cot(\theta) - i)^{2n+1}$, which we know must be equal to 0. We will get $0 = \sum_{r=1}^{2n+1} \binom{2n+1}{r} \cot^r(\theta)\, i^{2n+1-r} - \sum_{r=0}^{2n+1} \binom{2n+1}{r} \cot^r(\theta)\, (-i)^{2n+1-r} = \sum_{r=1}^{2n+1} \binom{2n+1}{r} \cot^r(\theta)\, [i^{2n+1-r} - (-i)^{2n+1-r}]$.

When r is odd, 2n+1-r is even, so the bracketed term will look like $i^{2k} - (-i)^{2k}$ where k is an integer, but this equals $(-1)^k - (-1)^k = 0$. Therefore the only terms that survive are the terms where r is even, so we can replace r with s ranging from 1 to n where s is 2r. We have

$$\sum_{s=0}^{n} \binom{2n+1}{2s} \cot^{2s}(\theta) \left[i^{2(n-s)+1} - (-i)^{2(n-s)+1}\right]$$

Lets replace s with n-s, and note the identity $\binom{2n+1}{2s} = \binom{2n+1}{2n+1-2s}$ (since we can choose 2s things in the same number of ways we can exclude 2s things). Lets call n-s k, then we have

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \cot^{2(n-k)}(\theta)\, [i^{2k+1} - (-i)^{2k+1}]$$

This last term is $i + i = 2i$ when k is even and $-i - i = -2i$ when k is odd. Since the whole thing equals 0, we can divide it through by 2i to get the following:

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \cot^{2(n-k)}(\theta)\, (-1)^k = 0$$

If $x = \cot^2(\theta)$ we can write the sum above as

$$\binom{2n+1}{1} x^n - \binom{2n+1}{3} x^{n-1} + \cdots + (-1)^n = 0$$

Note that the roots of the polynomial above are exactly the n numbers $\cot^2\left(\frac{m\pi}{2n+1}\right)$ for m an integer going from 1 to n. The same logic applies if m is an integer greater than n or less than 0, however this would give one of the same numbers – this can be argued from symmetry of the graph, or from the fact that it must be the case as $\cot^2\left(\frac{m\pi}{2n+1}\right)$ for m from 1 to n is always distinct as cot is increasing in that range, and an n-degree polynomial cannot have more than n roots (or else you could factor it more than n times by the factor theorem).

Now we know that the sum of roots of a polynomial is minus the second leading coefficient divided by the leading coefficient. Applying this to the polynomial above gives

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{\binom{2n+1}{3}}{\binom{2n+1}{1}} = \frac{\frac{(2n+1)!}{3!\,(2n-2)!}}{2n+1} = \frac{(2n)!}{6(2n-2)!} = \frac{(2n)(2n-1)}{6} = \frac{n(2n-1)}{3}$$

We now need two facts:

- For θ between 0 and $\frac{\pi}{2}$, $\sin(\theta) = \int_0^\theta \cos(x)\,dx$. Since cos(x) is always between 0 and 1 in that range, it means that $\sin(\theta)$ is between 0 and θ
- For θ between 0 and $\frac{\pi}{2}$, $\tan(\theta) = \int_0^\theta \sec^2(x)\,dx$ (you can write tan as $\frac{sin}{cos}$ and differentiate it using the quotient rule and verify that it is sec^2). Since cos(x) is always between 0 and 1 in that range, it means that $sec(\theta)$ is greater than 1 for all non-zero values in that range, and therefore $\tan(\theta) > \theta$

So $0 < \sin(\theta) < \theta < \tan\,(\theta)$ for θ between 0 and $\frac{\pi}{2}$. Since in $\sin(\theta) < \theta < \tan(\theta)$ they are all positive, taking the reciprocal of both sides flips the inequality, so $\cot(\theta) < \frac{1}{\theta} < \csc(\theta)$. We still have positive things, so we will square both sides to get $\cot^2(\theta) < \frac{1}{\theta^2} < \csc^2(\theta)$. However we can say something else, which is that $\cot^2(\theta) < \frac{1}{\theta^2} < 1 + \cot^2(\theta)$, as $\csc^2(\theta) = \frac{1}{\sin^2(\theta)} = \frac{\sin^2(\theta)+\cos^2(\theta)}{\sin^2(\theta)} = 1 + \cot^2(\theta)$.

Now $\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) < \sum_{m=1}^{n} \frac{(2n+1)^2}{m^2\pi^2} < \sum_{m=1}^{n} 1 + \cot^2\left(\frac{m\pi}{2n+1}\right) = n + \sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right)$

But we know $\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{n(2n-1)}{3}$, therefore $n + \sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{n(2n+2)}{3}$. Therefore we have the tight bound $\frac{n(2n-1)}{3} < \sum_{m=1}^{n} \frac{(2n+1)^2}{m^2\pi^2} < \frac{n(2n+2)}{3}$ for all positive integers n. Lets multiply through by $\frac{\pi^2}{(2n+1)^2}$ to get that $\frac{n(2n-1)\pi^2}{3(2n+1)^2} < \sum_{m=1}^{n} \frac{1}{m^2} < \frac{n(2n+2)\pi^2}{3(2n+1)^2}$. But notice that as n gets large, $\frac{n(2n-1)}{3(2n+1)^2}$ and $\frac{n(2n+2)}{3(2n+1)^2}$ will both approach $\frac{1}{6}$, as they are exactly $\frac{1}{6}$ save for the +1 and +2 and -1 terms which contribute a factor on the order of $\frac{1}{n}$. Therefore the sum $\sum_{m=1}^{n} \frac{1}{m^2}$ is bounded below and above by something that approaches $\frac{\pi^2}{6}$, and therefore the infinite sum is equal to exactly $\frac{\pi^2}{6}$.