We can do reverse percentages. Be careful: If I have some money and after increasing it by 20% I have $60, this does not mean that to find the original amount I subtract 20% of 60. What actually happens is that 60=x(1+0.20) so by some algebra x=50, but you would get x=48 if you did it the naïve way.

We can compound percentages: If I increase by a further 25% I have $75, if I then increase that by 20% I will have $90.

We can have shapes and angles. 360 degrees is a full rotation, 180 is a half rotation and 90 is a quarter rotation. We can have shapes like triangles and squares and hexagons and stuff. The angles inside a shape add up to $180(x-2)$ where x is the number of sides in the shape. The proof for the case that the polygon is convex (ie, has no dents or inward curves) is simple so we will give it here, but the general proof is deferred to level 4.

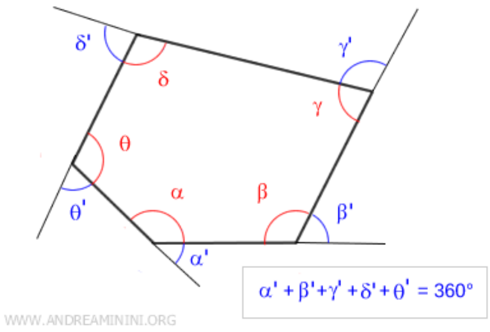


Image: Shows a diagram to demonstrate why internal angles + external angles = 180 * the number of sides, and why the external angles of a convex polygon sum to 360.

We note that the total of the red and blue is clearly 180 degrees times the number of verticies or sides, and that he sum of the blue angles is clearly 360 degrees as you could imagine dragging α' over to β' then those two together over to Y' and so on and then we will see that the total of the blues is clearly 360 degrees. The total of the reds is thus the difference between the total of the red and blue (180n) minus the total of the blue (360) which gives 180n-360 as required.

A regular polygon has all sides and angles the same.

Example: Suppose the angles in a regular polygon are 165 degrees each and we want to find how many sides are in the polygon.

Then we know the sum of the angles is equal to 165n where n is the number of sides, since there are n angles each with size 165. We also know this total is 180n-360 by the formula above. Therefore these must be equal to eachother. We have applied the general principle of seeking two different pieces of information and now we will combine them. 165n=180n-360 can be solved for n to get n=24.

If a triangle has an angle that is 90 degrees, ie a right angle, and the length of the two shorter sides are a and b, and the length of the longer side is c, then the length of the sides are related by $a^2 + b^2 = c^2$. This is the pythagorean theorem. I will show an example of how this works and then show a proof.

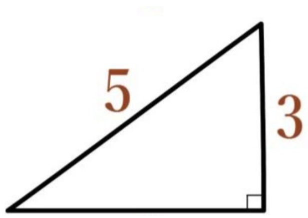


Image of a right angled triangle

Now we want the length of the bottom side. Lets call this x, then $3^2 + x^2 = 5^2$. This implies that $x^2 = 16$. It turns out that $x = 4$ works. $x = -4$ also works but the length is never negative.

The area is how much space the triangle fills. It turns out this is 6, since the area is half the base (4) times the height (3): The area of the orange and green parts in the image below together is the base times the height. The area of the orange triangle is half of that.

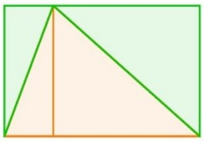


Image of a triangle inside a rectangle to show the area

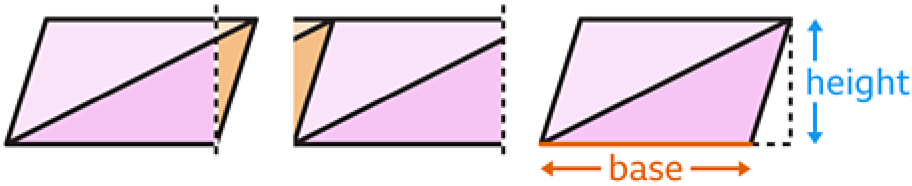


Image to show why this works when the triangle does not neatly fit into a rectangle.

As promised, here is a proof of pythagoras:

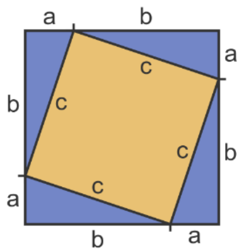


Image of a square with another square inside it

In the image above, there are four blue triangles with sides abc. The area of the large square is the area of the orange square plus four of the blue triangles. This is $4\left(\frac{1}{2}ab\right) + c^2$. The area is also equal to the square of the side length, ie $(a + b)^2$ which can be expanded to obtain $a^2 + 2ab + b^2$. Therefore we know that two things equal to the area of the large square are equal to eachother, which means that $a^2 + b^2 + 2ab = c^2 + 2ab$ so therefore $a^2 + b^2 = c^2$.

This theorem can be generalized: If I move 3 meters forward and 4 meters to the right then the square of how much distance I will have covered if I draw a line from my initial to final positions is $3^2 + 4^2$ meters. Then if I move 12 meters up, since the up direction is perpendicular (ie makes a right angle) to any movement I do along the ground, it means that the square of the distance I have moved in total will be $3^2 + 4^2 + 12^2$, so it just so happens I will have moved a total of 13 meters if I had gone in a straight line. Therefore, in the three perpendicular directions (up, forward and right), I can add the squares of the distance each of my “coordinates”, ie my height, rightness, and forwardness have changed to get the square of my overall distance. Alternatively I can find that long the diagonal of a 3*4*12 brick is 13 in length by the same principle

The volume is the analagous thing to area for 3 dimensional things.

The definition of π (which is about 3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117067982148086513282306647093844609550582231725359408128481117450284102701938521105559644622948954930381964...) is the distance around a circle that is 1 long. The radius of a circle (r) is half its length (diameter). Then the area of a circle is given by $\pi * r^2$. The proof is reserved for level 4. Note that the area is clearly a multiple of r^2 as if we scale the circle the area changes related to the square of how much we scale the circle by, as it scales by that amount by the base and the height directions.

Every number can be uniquely factored into prime, for example $630 = 2 * 3 * 3 * 5 * 7$. This is because you can decompose the number until you are down to primes. The proof that this can be done uniquely is reserved for level 4.

You can also have sets of things. As an example, let A={2,3,5,7} be a set and B={1,3,9} be a set. Then we can do $A \cup B$ to mean "A union B" which means the set of everything in A or B, which in this case would be {1,2,3,5,7,9}. $A \cap B$ means "A intersection B" which in this case would be {3}. The notation $3 \in A$ means 3 is an element of A. The notation $\{3,5\} \subseteq A$ means {3,5} is a subset of A, ie it is contained in A. We can do venn diagrams.

$\mathbb{N}$ is the symbol for the set of positive integers or natural numbers. Convention about if this includes 0 varies. $\mathbb{Z}$ is the symbol for the set of all integers (not necessarily positive). $\mathbb{Q}$ is the symbol for the set of rational numbers (numbers that can be represented by fractions of integers, these are not all numbers as we prove numbers like π and some square roots are not rational). $\mathbb{R}$ is the symbol for the set of all real numbers. A\B is typically used to mean the set of stuff in a set A but not a set B.

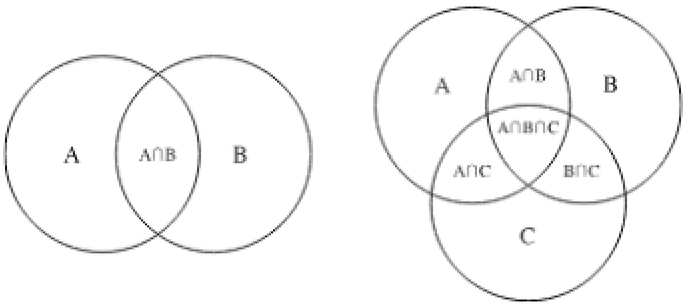


Image of a venn diagram

To compute the lowest common multiple or highest common factor of two numbers you can put the prime factors into a venn diagram and multiply together everything in the diagram and everything in the intersection respectively.

It is possible to solve simaltaneous equations. Suppose we have $3x + 4y = 46$ and $2x + y = 19$, then we can rearrange any equation to get one variable, in this case that $y = 19 - 2x$. Then we can put that back into the **other** equation to get $3x + 4(19 - 2x) = 46$, and we can solve this for x to get $x = 6$, then by either equation, say the second one $2 * 6 + y = 19$, so $y = 7$. Such systems of equations can possibly have one single solution, more than one, infinitely many, or zero solutions.

We can also solve inequalities. For example, if we want to find the set of values of x such that $2x + 3 < 5$, we can rearrange to get $2x < 2$ and then $x < 1$. It is ok to multiply or divide inequalities by positive real numbers or add or subtract real numbers from both sides, but if you multiply both sides by a negative real number you must flip it. Ie, if we have $3 - x < 2$ this implies $-x < -1$, so because minus x lies to the left of -1 on the number line, we have $x > 1$.

On inequalities:

- Notice that if x is positive, $\frac{1}{x}$ is smaller the larger x is – The more slices you cut a cake into the smaller each slice is. This means that if $a < b, \frac{1}{a} > \frac{1}{b}$, so if you take the reciprocal of both sides of an inequality where you know both sides are positive (or equivalently, both negative) you need to flip it. Note that one divided by a number is called the **reciprocal** of that number.

- If x is positive x^2, or any positive power of x in fact, gets bigger as x does, which I'm sure does not need much convincing. Therefore if both sides of an inequality are positive we can safely raise them both to a positive integer power (in fact any positive power, but we have not defined this yet, and it is less obvious that it is increasing but after level 3 we could argue that with differentiation)

Inequalities can be strict or not strict. $<$ means less than and $\leq$ means less than or equal to and similarly for greater than.

We can plot equations on graphs. Play around with desmos to see how this might work. But we can have a graph like $y = 3x + 5$, then for every change in x by 1 y will change by 3, and when x is 0 y will be 5, so we have a straight line with "slope" 3 that intersects the y axis (the line x=0) when y=5, and by some algebra we find that y=0 when $x = -\frac{5}{3}$. The image below shows the graph.

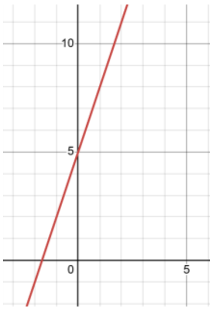


Image: Graph of y=3x+5

The simaltaneous equations above can be interpreted as graphing 2 lines and finding where they meet.

We can find a line between 2 points. Lets say we want to find an equation for the line that passes through (2,3) and (5,7) where this means the point x=2, y=3 and the point x=5, y=7. We know that any line equation has the form $y = mx + c$. Therefore $3 = 2m + c, 7 = 5m + c$. Solving for m and c gives that $m = \frac{4}{3}, c = \frac{1}{3}$. So the equation is $y = \frac{4}{3}x + \frac{1}{3}$. Rearrangeing this gives $3y - 4x - 1 = 0$.

A quadratic is an expression that looks like $Ax^2 + Bx + C$, ie it involves squaring a variable or multiplying it by a number or adding it to a number but that is it. They can be factorized and solved that way. By some algebra, $x^2 - 5x + 6 = 0$ implies that $(x-2)(x-3) = 0$, so x is 2 or 3. Therefore the solutions are x=2 and x=3, as $(x-2)(x-3)$ is 0 exactly when either x-2 or x-3 is 0. The solutions to $x^2 - 5x + 6 = 0$ are called the **roots** of $x^2 - 5x + 6$ but such equations do not always factor easily. An example is $x^2 + 2x + 2 = 0$.

Note that, as with the example $x^2 - 5x + 6 = (x-2)(x-3)$, it is generally true that $(x-A)(x-B) = x^2 - (A+B)x + AB$, as we can just expand it. This tells us that we should look for numbers that add up to minus the coefficient (ie the thing multiplying) of x and times up to the term not depending on x (ie the constant term). These may not be positive, for example in $x^2 - x - 2 = 0$ the numbers I am looking for are 2 and -1. Usually you want to find these by trial and error, otherwise you want to use a formula which I will show later.

We all have an intuitive idea of what probability is. You can think of "The probability of X happening is 0.6" to mean that if you keep doing trials, you would expect X to happen on about 60% of trials. We can multiply probabilities, X happens twice in a row with probability 0.6*0.6, 3 times in a row with probability 0.6*0.6*0.6, in general if an event has probability p, it happens n times in a row with probability p^n. p is always between 0 and 1 for this to make sense. Also, the probability of something not happening is 1-p: An analogy is if there is a 70% chance of rain, this translates to p=0.7, then the chances of not rain is 30%, which translates to p=0.3, which indeed is the same as 1-0.7.

You can also add probabilities provided events are mutually exclusive. What mutually exclusive means is that they cannot both happen. For example, a dice cannot land on both a 1 and a 2. The probability it lands on a 1 is $\frac{1}{6}$ and the probability it lands on a 2 is also $\frac{1}{6}$. So the probability it lands on a 1 or a 2 is $\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$. However, you cannot say that the chance of getting a 1 on either the first roll or the second roll is $\frac{1}{3}$, since this is false. It is actually the sum of the probability that you get it on the first roll with the probability that you do not get it on the first roll but get it on the second roll. These are actually mutually exclusive. You will end up getting $\frac{1}{6}+\frac{5}{6}*\frac{1}{6}=\frac{11}{36}$. We can multiply probabilities for the and because the events do not depend on one-another, ie there are independent.

Probability is an area with a lot of mathematical misconceptions that I have seen from teachers and wikipedia articles. Ask yourself what is the probability that you get the same number on a dice twice in a row. Naively you might say that it is $\frac{1}{6}*\frac{1}{6}=\frac{1}{36}$, but this is false, as what happens is that what you get on the first roll does not really matter and then what you get on the second roll merely has to be the same, and regardless of what you got on the first roll this has probability $\frac{1}{6}$. Also, in a random string of digits, the probability of seeing 6 of any digit in a row is about 10 times higher than the probability of seeing 6 of a specific digit in a row, since there are 10 times as many possibilities – this mistake used to be on the wikipedia article for the feynman point in pi.

In statistics, if I have a list of data, there are a few ways to try to get an average or an idea of the spread. Lets say my data is 1,3,4,5,6,6,6, then the number that occurs in the middle of the sorted data (**median**) is 5, the number that occurs most often (**mode**) is 6, the result of adding them and dividing by the number of data (**mean**) is about 4.43, and the **range** (largest minus smallest) is 5. We can also calculate ranges excluding the most extreme 25% or 5% of data on either side, there are different conventions but the idea is to have extreme values (outliers) be excluded.

In a right angled triangle with another angle x, we have the long side (h), the other side that touches the angle (a) and the remaining side o. Then the ratio $\frac{o}{h}$ is the **sine** of the angle x, which we write sin(x). Similarly we write $\frac{a}{h}$ as cos(x) (**cosine**) and $\frac{o}{a}$ as tan(x) (**tangent**). These are trigonometric functions

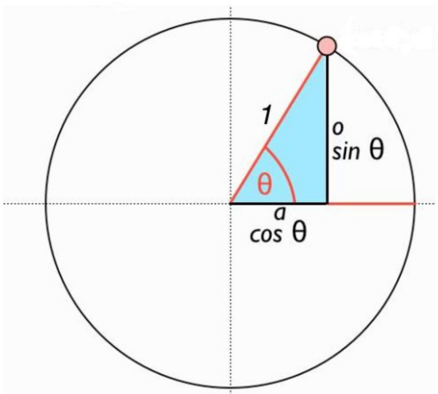


Image of a triangle in a circle to show another interpretation of this

In the image below we see that if h=1 then cosθ and sinθ are exactly our x and y coordinates when we move by an angle θ along the circle anticlockwise from the point (1, 0).

Triangles are similar if they have the same angles, meaning you can transform one into the other by scaling, rotating, reflecting and moving it. They are congruent if you impose also that their sides are the same.

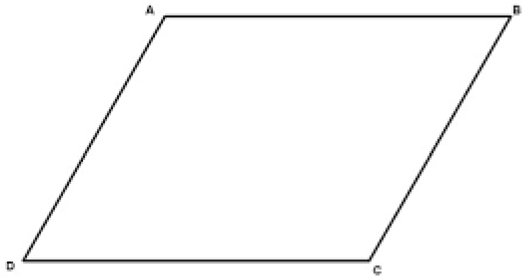
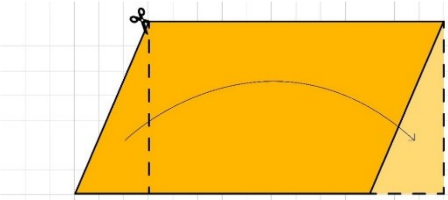


Image of a parallelogram

A parallelogram is a 4 sided shape with 2 sets of parallel sides, parallel meaning they point the same way. The area of a parallelogram is just the base times the height, as you can turn it into a rectangle by cutting and moving parts as in the image below.



A trapezium is a 4 sided shape with one set of parallel sides.

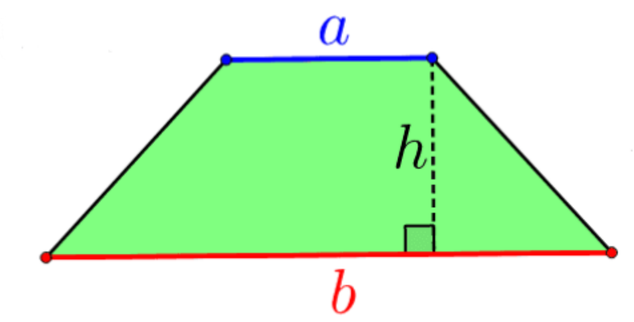


Image – To show how we are labelling the sides

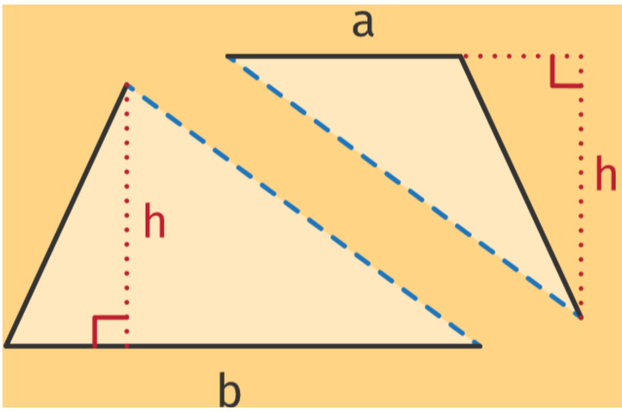


Image – Splitting a trapesium into 2 triangles to show why the area is $h\left(\frac{a+b}{2}\right)$.

Now we can write $\sin^{-1}(x)$, or the notation I will use is $\arcsin(x)$ to mean the angle from -90 to 90 degrees such that its sine is equal to x. We can do the same for $\arccos(x)$ where we pick the angle from 0 to 180 degrees such that its cosine is equal to x, and we can do the same for $\arctan(x)$ to pick the angle from -90 to 90 degrees such that its tangent is equal to x. The circle interpretation is the way this stuff is extended to negative angles.

Note that in the extreme cases, we have the following:

0 degrees – By the circle interpretation, $\sin(0°) = 0$, $\cos(0°) = 1$, and we have a useful formula: $\tan(x) = \frac{o}{a} = \frac{\frac{o}{h}}{\frac{a}{h}} = \frac{\sin(x)}{\cos(x)}$ which we can use to show that $\tan(0°) = 0$.

An equilateral triangle (all sides and angles the same) will have all its angles 60 degrees since all three of its angles must add up to 180 degrees by a formula above. If we take half of it we will therefore get a triangle with angles 30, 60 and 90 degrees. Say we start with an equilateral triangle with side lengths 1 with one side horizontal then cut it so we only have the left side. Then the base will have length $\frac{1}{2}$, then the side that was originally part of the equilateral triangle that is on the left will be unchanged and have length 1. Lets call the remaining side length x. By pythagoras, $x^2 + \left(\frac{1}{2}\right)^2 = 1^2$, so by a bit of rearranging we have that $x^2 = \frac{3}{4}$

Definition: The square root of a positive number ($\sqrt{x}$) is the positive number that gives x when you square it. We see that $x = \sqrt{\frac{3}{4}}$. Note that the square root of a number x can be written as $x^{\frac{1}{2}}$ because by the usual rules of powers $x^{\frac{1}{2}}x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$. The fact that we can do this is something we will make more rigorous in level 4 where we will define powers formally for real numbers and show that they satisfy these rules and we will know that we have a proper definition, but for now you should assume that the rules of exponents are consistent and that we can “abuse” them in this way for real numbers. Now note also that $x = \sqrt{\frac{3}{4}} = \left(\frac{3}{4}\right)^{\frac{1}{2}} = \frac{3^{\frac{1}{2}}}{4^{\frac{1}{2}}} = \frac{\sqrt{3}}{2}$ by exponent rules and the fact that the square root of 4 is 2. Now we have all the sides of our triangle and can therefore deduce the following:

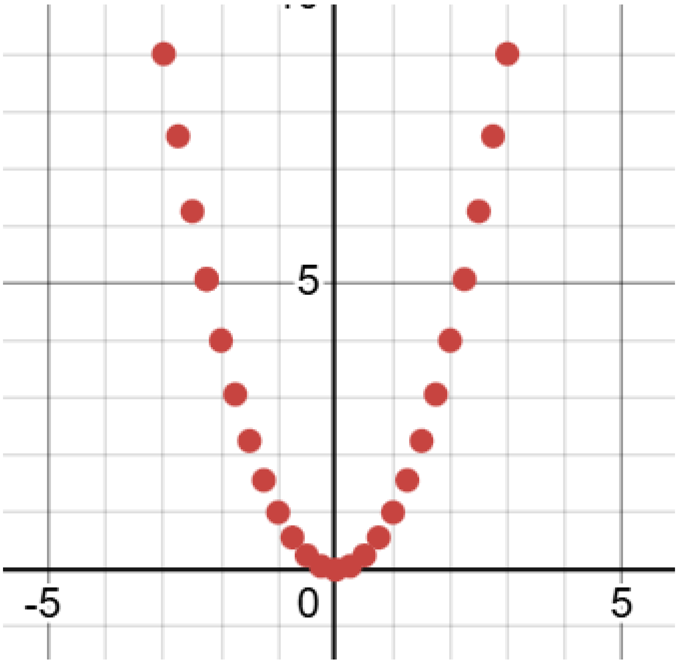
- $\sin(30°) = \frac{1}{2}$
- $\cos(30°) = \frac{\sqrt{3}}{2}$
- $\tan(30°) = \frac{1}{\sqrt{3}}$
- $\cos(60°) = \frac{1}{2}$
- $\sin(60°) = \frac{\sqrt{3}}{2}$
- $\tan(60°) = \sqrt{3}$

Now in a 45, 45, 90 triangle the two sides touching the right angle are the same length, and by pythagoras the remaining side has length $\sqrt{2}$. Therefore we have

- $\cos(45°) = \frac{1}{\sqrt{2}}$
- $\sin(45°) = \frac{1}{\sqrt{2}}$
- $\tan(45°) = 1$

We can make functions of x. For example if $f(x) = x^2$ then $f(3) = 9$. A function is something, usually denoted f, that takes in one value and returns exactly one value. The set of values such that a function is defined is called its domain and the set of values it can return is called its codomain or range. In the case above we can define $f(x) = x^2$ with domain real numbers and range non-negative numbers (since the square of a number is never negative as whether you square zero or a negative thing or a positive thing you get a non-negative thing). It is best practice to specify the domain of a function when you define it.

We can compute a few values of x^2 and put these on a graph, and this is what that looks like.



We can now connect these to get a continuous curve

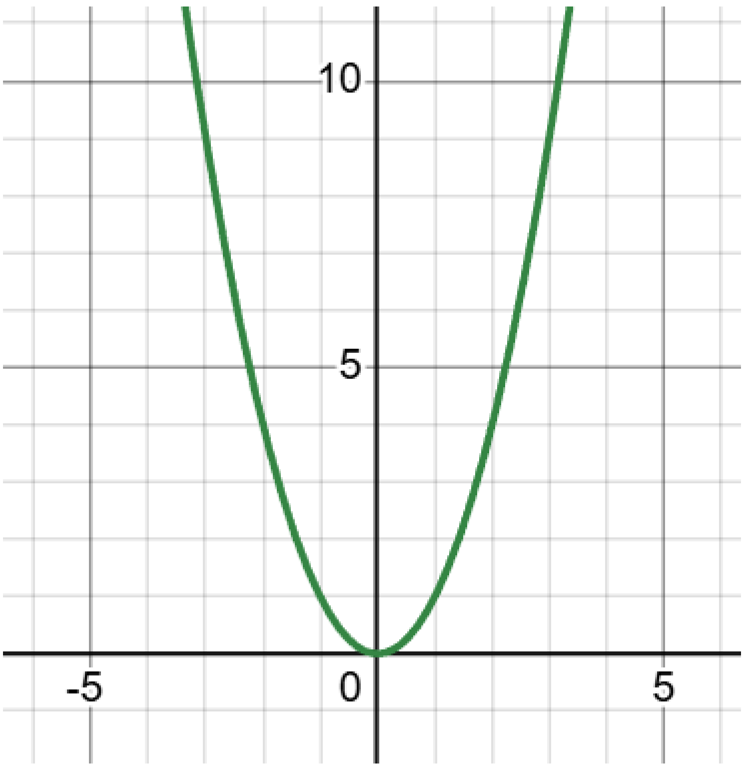


Image: Graph of $y = x^2$

This shape is called a parabola. In general you can compute some values and try to connect the points in order to graph any reasonablve function. For example,

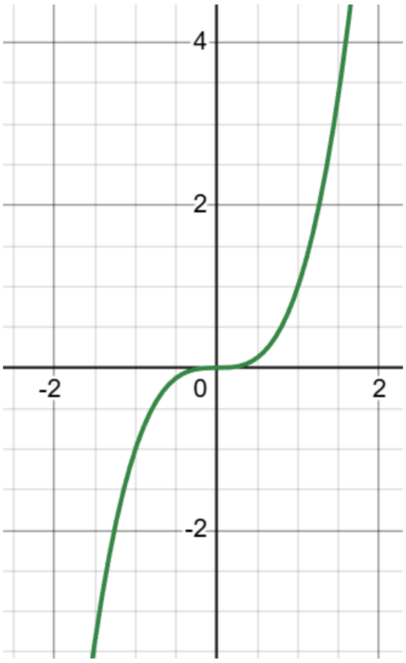


Image: Graph of $y = x^3$

Once you have seen these graphs once you will know the general shape and will eventually be able to sketch them without having to compute values and connect the dots. We will see how this works as we go on.

We can transform objects, in particular graphs. Here are some ways we can do that:

- Shifting (Translation): This preserves lengths and angles
- Reflection about a line: This preserves lengths and angles but not orientation
- Enlargement by an axis: This does not preserve lengths or angles
- Enlargement by both axes: This preserves angles but not lengths

If $y = f(x)$ and we know what that graph looks like, then we can work out what the graph $y = f(x + a)$ must look like for any choice of a. This is because the value of the function $f(x + a)$ at any x value is just the same as the value of $f(x)$ at the x value a distance of a units away.

Examples:

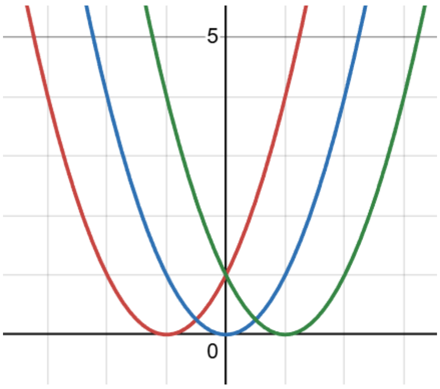


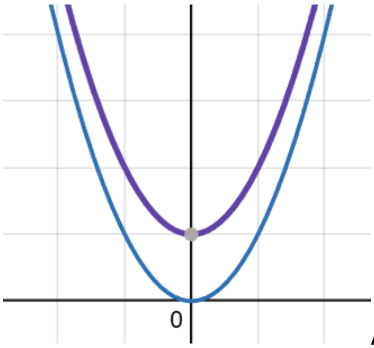
Image of graphs

Red graph is $y = (x + 1)^2$

Blue graph is $y = x^2$

Green graph is $y = (x - 1)^2$

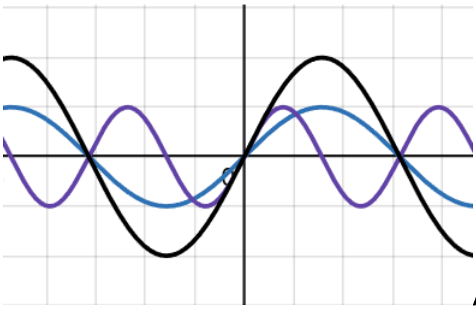
Note that $y = f(x) + a$ shifts upwards



Another image of graphs

Blue is $y = x^2$, Purple is $y = x^2 + 1$

We can also have $y = af(x), y = f(ax)$, I will show both of those shortly, then after that one can combine transformations to get graphs of $y = af(bx + c) + d$



Another image of graphs

In the image above, blue is $y = f(x)$, purple is $y = f(2x)$ (For each x value we need to find f of double that so it moves along twice as fast), and black is $y = 2f(x)$ (Each y value is doubled from blue).

Those are all the simple transformations on graphs.

When we learn how to complete the square it will follow how you can use this to sketch any graph of the form $y = ax^2 + bx + c$.

Lets now sketch the graphs of $\sin(x)$, $\cos(x)$ and $\tan\,(x)$.

As you move around a circle, it is easy to see that your x and y coordinates which are cosine and since of the angle respectively if you plot them on the y axis with time on the x axis will look like this:

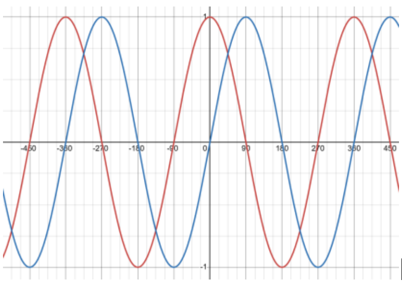


Image: $y = \sin\,(x)$ (red), $y = \cos\,(x)$ (blue)

As usual it is possible to transform these graphs as above.

Now to find $\tan(x) = \frac{\sin(x)}{\cos\,(x)}$ we want $\frac{y}{x}$ as we move alone the circle, ie the slope of the line from the origin to that point on the circle. Here is a graph of how that changes:

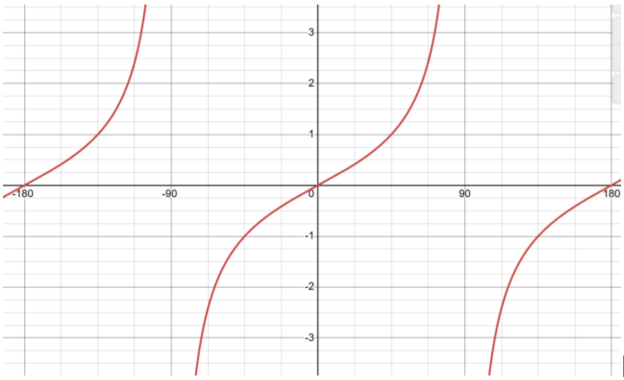


Image: Graph of $y = \tan(x)$

Again it is possible to do transformations on this.

Now notice how in each of these graphs of functions, all possible y values (outputs of the function) can be achieved by infinitely many x values. This moves me to a very important general point:

Things to be careful of

Do not cancel zeroes, for example 0*3=0*5 does not imply 3=5. If cancelling any factor, justify that it is not zero. If you have ac=bc and cancel the c factor, what you should do is deal with the c=0 case separately, ie ac=bc implies that either a=b OR c=0.

Do not cancel squares and square roots when the square is inside the square root unless you justify that the thing being squared inside the square root is a non-negative real number. For example, $\sqrt{(-3)^2} = \sqrt{9} = 3$ which is not equal to -3. This is because square root is defined the positive square root. If it were a proper inverse of the square function, it would have to take two values (for example $\sqrt{9} = 3, -3$) and then it would not be a function, by definition.

If you square both sides of anything, you must either be sure that it is not the case that one side is non-zero but minus the other side (Since then they could be non-equal but have equal squares, both sides being non-negative reals would suffice), or check after for extraneous solutions, for example,

$$x - 1 = 3$$

(Clearly x=4, but for the sake of example I will square both sides to show what goes wrong)

$$(x-1)^2 = 9$$

$$x^2 - 2x + 1 = 9$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

So the extraneous solution x = -2 arises. It is true when squaring both sides that each step is implied by the previous, but that does not mean that each step is equivalent to the previous.

The reason is essentially that more things can be squared to give 9 than just 3. If a function is one-to-one from the domain to the range, you may apply it to both sides or cancel it from both sides, but otherwise it requires justification.

Just like how squaring both sides can cause you to gain solutions, the opposite is true. For example, x=-3 is a solution of

$$x^2 = 3^2$$

But cancelling the squares would give x=3. When cancelling squares on both sides or cancelling squares on one side or square rooting the other, you should put a $\pm$ on one of the sides to ensure no loss of information. This symbol means "plus or minus".

Also, note that trigonometric functions are one-to-many over the real numbers, so similar problems arise. If $y = arcsin(x)$, it is true that $x = sin(y)$ meaning $x = sin(arcsin(x))$ when x is from -1 to 1 inclusive. However, if $y = sin(x)$ it is not necessarily the case that $x = arcsin(y)$, because consider what happens if $x = 180°$: arcsin is defined to take the value from $-90°$ to $90°$, so $arcsin(sin(\pi)) = 0$ which is a counterexample, kind of like how the square root of a negative number squared is a positive number since square root is defined to output positive values.

I guess my point is you need to carefully justify cancelling functions that you think are inverses of each other or applying or cancelling one-to-many functions to both sides of an equation.

Lets do an example where we are being careful to really drive this home.

We want to solve for x in $\sqrt{3x-5} - \sqrt{x+6} = 1$ for real values of x.

Note that square roots of real numbers are always positive, so here is one way to do that.

Lets square both sides, remembering to check each solution at the end to make sure it is not extraneous. This gives $3x - 5 + x + 6 - 2\sqrt{(x+6)(3x-5)} = 1$, since we know it is true that $\sqrt{x+6}\sqrt{3x-5} = \sqrt{(x+6)(3x-5)}$ by exponent rules. Rearranging this a little bit gives

$$4x = 2\sqrt{(x+6)(3x-5)}$$

$$2x = \sqrt{(x+6)(3x-5)}$$

Now lets square both sides again

$$4x^2 = (x+6)(3x-5) = 3x^2 + 13x - 30$$

$$x^2 - 13x + 30 = 0$$

$$x = 3, x = 10$$

But one can check by plugging in $x = 3$ to both sides of the equation that it is extraneous and only the solution $x = 10$ works. The problem was when we squared both sides the first time. Squaring both sides the second time was fine because square roots of real things are always positive by convention so each square of a thing we know is positive could only have come from one possible input.

A sequence is a list of numbers. If I give you an expression for the n'th term such as $n^2 + 1$ you can generate your sequence by putting 1, 2, 3, etc into the expression. For the above expression the resulting sequence would be 2, 5, 10, 17, 26, 37, etc

Now if I give you a sequence where the rule to get from one term to the next is something simple, such as adding 4, you can reverse engineer the rule by pattern spotting. For example, a sequence may be given by 1, 5, 9, 13, 17, 21, ... A common mistake I've seen is to say that "$n + 4$" is the expression that returns the n'th term in this sequence. But that is the rule to get from one element to the next which is not the same thing The sequence generated by $n + 4$ would actually be the sequence 5, 6, 7, 8, 9, ... For a sequence where each term is, say 4, more than the previous, trying 4n gets us close – this gives us the sequence 4, 8, 12, 16, 20,... and each term is 3 too large, so 4n-3 gets us the desired sequence 1, 5, 9, 13, 17, 21, ... I actually remember that this was the first question on my Maths GCSE.

We can go beyond square roots. We can have cube roots of numbers - $\sqrt[3]{x}$ is the cube root of x, ie the real number y such that $y^3 = x$. These are unique because for positive x, x^3 is increasing, and for negative x, $(-x)^3 = (-x)(-x)(-x) = -(x^3)$ since there are an odd number of minuses and pairs of minuses cancel, so x^3 is increasing for negative x as well, and from this and the graph $y = x^3$ you see that it is a one-to-one function that can map any real number to any other real number so it does have a proper inverse, ie the cube root. Similar to with square roots, $\sqrt[3]{x} = x^{\frac{1}{3}}$

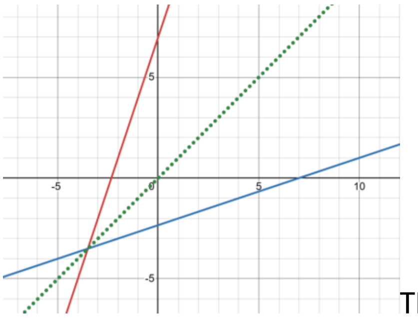
Inverse functions are a more general concept – a function you get by "undoing" another function. We have seen examples like arcsin as an inverse of sin, but lets see how you can find inverse functions in general. If $y = f(x)$ we want to find x as a function of y, and this function we call $x = f^{-1}(y)$. As an example, if $f(x) = 3x + 7$ here is how we can find an inverse (whenever the function is one to one from its domain to its range):

$$y = 3x + 7$$

$$3x = y - 7$$

$$x = \frac{y - 7}{3}$$

So our rule for f^{-1} is that $f^{-1}(x) = \frac{x-7}{3}$, where I have written x in place of y in $f^{-1}(y) = \frac{y-7}{3}$. Lets see geometrically what this does:



The image here shows $f(x)$ in red and $f^{-1}(x)$ in blue. We note that if y is an inverse function of x it is essentially like x as a function of y, so it is essentially like swapping the axes around, which has the geometric effect of reflecting about the green dotted line $y = x$.

We also can have compound functions where we essentially do 1 then the other. Lets define two functions $f(x) = 3x + 7$ and $g(x) = x^2 + 1$, then:

- $f\big(g(x)\big) = f(x^2 + 1) = 3(x^2 + 1) + 7 = 3x^2 + 10$
- $g\big(f(x)\big) = g(3x + 7) = (3x + 7)^2 + 1 = 9x^2 + 42x + 50$

Notice that they are not the same.

Now lets do a problem involving powers to show how important it is to put everything on the same base. We want to simplify $\frac{3^5 * 27^{\frac{1}{4}}}{\sqrt[3]{9}}$. First, we spot that $\sqrt[3]{9}$ is $9^{\frac{1}{3}}$, then we notice that our bases are 3, 9 and 27, which are $3^1, 3^2, 3^3$ respectively, so we write $\frac{3^5 * (3^3)^{\frac{1}{4}}}{(3^2)^{\frac{1}{3}}} = \frac{3^5 * 3^{\frac{3}{4}}}{3^{\frac{2}{3}}} = 3^5 * 3^{\frac{3}{4}} * 3^{-\frac{2}{3}}$ by a rule of powers, and if we multiply things with the same base we can add powers, we know how to add fractions from level 1, and we end up with $3^{\frac{61}{12}}$ as our answer.

Example: Suppose we want to solve for x in $2^{3x-5}4^{1-x} = 1$. Then again we want to get everything onto the same base, and in this case 2 is the obvious choice. We can write $2^{3x-5}(2^2)^{1-x} = 1$ and use rules of powers to get $2^{3x-5}2^{2(1-x)} = 1$ so $2^{3x-5+2(1-x)} = 1$. Since something that is not 1 to the power of a real number being 1 implies the thing in the power is 0 (since if we increase the power the result strictly increases so the result can only be 1 once), we therefore have $3x - 5 + 2(1 - x) = 0$ which after simplifying implies x=3.

Example: We want to simplify $\sqrt{80} + \sqrt{125}$. We do this by fully factoring the square roots into primes. This gives $\sqrt{5 * 2^4} + \sqrt{5^3}$. We now pull out all the squares we can, ie for powers that are larger than 2, we split them until each power is at most 2: $\sqrt{5 * 2^2 * 2^2} + \sqrt{5 * 5^2}$. Now we take the squares out of the square roots using rules of powers: $\sqrt{2^2}\sqrt{2^2}\sqrt{5} + \sqrt{5^2}\sqrt{5}$. Since we are assuming that we want the positive square root, we can cancel the square root with the square. Then $2 * 2 * \sqrt{5} + 5 * \sqrt{5} = 9\sqrt{5}$ is our final answer

Example: $\frac{3.2+7.5*9.1}{6.4-3.7}$ is a fraction where everything is rounded to one decimal place and we want to find the largest and smallest possible value it can take. Note that since something is larger the smaller the denominator is whenever the denominator is positive, to maximize it we want to maximize the numerator and minimize the denominator. As an example, 3.2 is rounded to 1 decimal place so its possible range of values is 3.15 to 3.25. So the maximum value of the fraction is

$$\frac{3.25 + 7.55 * 9.15}{6.35 - 3.75} \approx 27.82$$

As to minimize the denominator we minimize the parts we are adding and maximize the parts we are subtracting. And the minimum possible value is

$$\frac{3.15 + 7.45 * 9.05}{6.45 - 3.65} \approx 25.20$$

Now lets go back to quadratic equations, ie equations like $x^2 + ax + b = 0$ for varying values of a and b. Note that $(x + c)^2 = x^2 + 2cx + c^2$ by simple algebra, so any expression of the form $x^2 + 2cx + d$ just requires you to add or subtract something and then it will be exactly $(x + c)^2$. So what we can do is $x^2 + 2cx + d = x^2 + 2cx + c^2 - c^2 + d$ since adding $+c^2 - c^2$ does nothing, and therefore we have that $x^2 + 2cx + d = x^2 + 2cx + c^2 - c^2 + d = (x + c)^2 + d - c^2$. In practice:

$x^2 + 4x + 5$ has $2c = 4$ so $c = 2$ and $d = 5$ so $x^2 + 4x + 5 = (x + 2)^2 + 5 - 2^2 = (x + 2)^2 + 1$. This is called completing the square

Now lets use this to solve an equation that we may not be able to factor easily: $x^2 - 7x + 12 = x + 1$. First, move everything to one side, so we have $x^2 - 7x + 11 = 0$. Completing the square (we solve for c and d, $c = -\frac{7}{2}, c^2 = \frac{49}{4}, d = 11, c^2 - d = \frac{5}{4}$ and so $x^2 - 7x + 11 = \left(x - \frac{7}{2}\right)^2 - \frac{5}{4}$, so we have that $\left(x - \frac{7}{2}\right)^2 - \frac{5}{4} = 0$ so $\left(x - \frac{7}{2}\right)^2 = \frac{5}{4}$. Therefore $x - \frac{7}{2} = \pm\sqrt{\frac{5}{4}} = \pm\frac{\sqrt{5}}{\sqrt{4}} = \pm\frac{\sqrt{5}}{2}$, so $x = \frac{7}{2} \pm \frac{\sqrt{5}}{2}$. However, this does not always work: For $x^2 + 4x + 5 = 0, (x + 2)^2 + 1 = 0$ by the example above so $(x + 2)^2 = -1$ but as discussed no real number squared is -1.

Now we will derive the quadratic formula by taking the example above. Set $ax^2 + bx + c = 0$ and seek a formula for x. Note that on most school tests, you just need to be able to use the formula, not re-derive it like we will do here. Lets do some algebra.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Assuming a is not 0, or else this is simple to solve. Now lets complete the square

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$x + \frac{b}{2a} = \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} = \pm\sqrt{\frac{b^2}{(2a)^2} - \frac{4ac}{4a^2}} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Therefore, rearranging, $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Note that when solving an equation using this formula, a lot depends on the form of the number inside the square root $b^2 - 4ac$. If this is a square number, we will have nice solutions, otherwise we will not, and if it is negative we will not have real number solutions.

There is also a convention sometimes used with the $\pm$ symbol where if it appears twice in an equation it may be the case that they should either be both + or both -, but this is not always true. However, if the symbols $\pm$ and $\mp$ are both used in an equation or expression then they have strictly opposite signs.

Aside from deriving the quadratic formula, completing the square has another important use which is sketching graphs of quadratics. I will give some examples. We start with the typical parabola $y = x^2$. Now suppose we want to sketch the graph $y = x^2 + 4x + 5 = (x + 2)^2 + 1$. We can sketch the graph $y = (x + 2)^2$ – we just have to take $y = x^2$ and shift it 2 to the left as instead of asking "what do we get when we square x" we are asking "what do we get when we square the thing 2 to the right of x on the number line" – recall graph transformations. Then we add 1 so we shift the graph 1 up. Here is what that looks like:

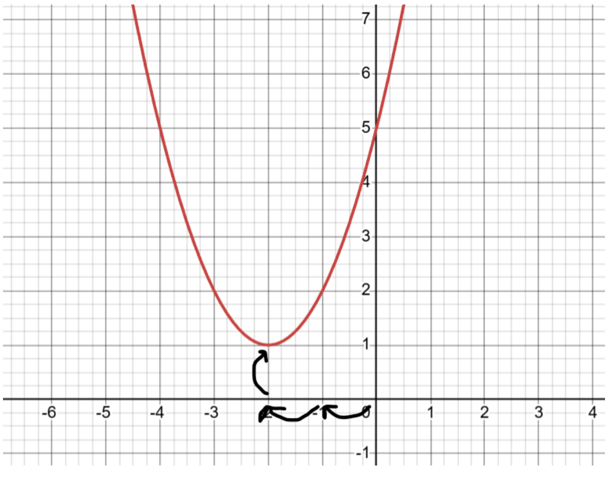


Image: Graph of $y = x^2 + 4x + 5$

By this method we can sketch any graph of the form $y = x^2 + Ax + B$. We can also sketch something like $y = 4x^2 + 17x - 3$: We can divide through by the leading coefficient of 4 to get $x^2 + \frac{17}{4}x - \frac{3}{4}$ which we can sketch with the above method and then just scale it up by 4 in the vertical direction after.

There is another thing we should notice. Consider a quadratic such as $x^2 - 7x + 10$, in which if we consider the equation $x^2 - 7x + 10 = 0$ this has roots/solutions (2 and 5 – can be found by factoring or using the formula, if you look for numbers that add to 7 and multiply to 10 you will see that 2 and 5 work). What this means is that if we sketch the graph $y = x^2 - 7x + 10$ it should cross through (2, 0) and (5, 0). Indeed this is exactly what happens.

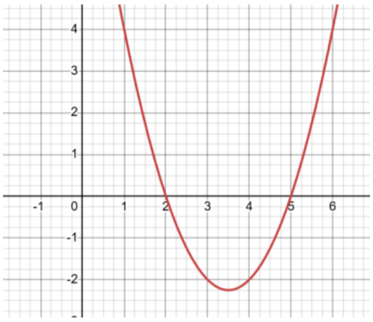


Image: Graph of $y = x^2 - 7x + 10$

Now we can solve inequalities with terms like x^2. Lets do an example.

Lets find all x such that $x^2 + 11x + 6 < 4x$. First, lets move everything to one side so that we have 0 on one side. This gives $x^2 + 7x + 6 < 0$. Now lets solve $x^2 + 7x + 6 = 0$. We can factor it – looking for 2 numbers that add to -7 and multiply to 6 we see that -1 and -6 work so our factorization is – being careful to get the minus signs all straight – is $\big(x - (-1)\big)\big(x - (-6)\big) = (x + 1)(x + 6)$. Therefore the solutions are $x = -1$ and $x = -6$. By previous work, we know that the graph of this will look like a parabola, and it will touch the x-axis exactly at the points -1 and -6. Here is an image of this:

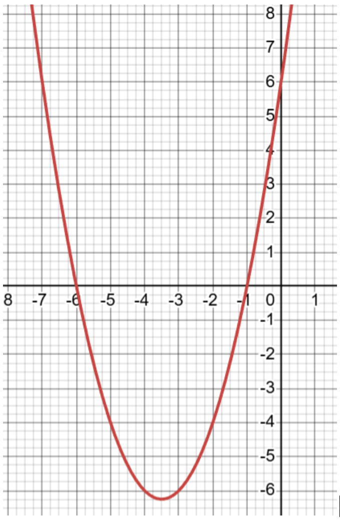


Image: graph of $y = x^2 + 7x + 6$

We see that -1 and -6 are where this crosses the x-axis. Therefore, the solution to $x^2 + 7x + 6 < 0$ is exactly when $-6 < x < -1$. Note that the inequality is $x^2 + 7x + 6 < 0$ and not $x^2 + 7x + 6 \leq 0$, so the points -1 and -6 are not solutions, and the solution is not $-6 \leq x \leq -1$. Read test questions carefully for this.

Now lets do a simpler proof involving algebra. Lets prove that if x is an integer then $x^2 + x$ is even. We factor $x^2 + x$ as $x(x + 1)$. If x is even this is an even number times an integer which is even. If x is odd then this is an integer x times x+1 which is even and thus even.

Note that an equivalent statement to the proposition above is that $\frac{n^2+n}{2}$ is an integer for every integer n, however we will now show this in a more elegant way. For the first few integer values of n, $\frac{n^2+n}{2}$ gives the sequence 1, 3, 6, 10, 15, 21, 28, ... But notice this is just 1, 1+2, 1+2+3, 1+2+3+4, ... So lets prove that this pattern always holds.

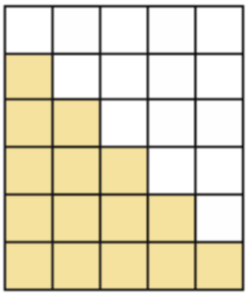


Image of a visual proof of the fact above

Notice that there are 1+2+3+4+5 yellow squares, in general we could make an analagous image for n more than 5. But notice that the number of yellow squares is half the total number of squares (it is easy to see this), but that is the number of rows (6, or in general n+1), times the number of columns (5, or in general n). Therefore, 1+2+3+...+n is the number of yellow squares which is half of the total number of squares which is half of n(n+1) so we get the formula $1+2+3+\cdots+n=\frac{n(n+1)}{2}$. As an example, the sum of every number up to 100 is $\frac{100(100+1)}{2}=\frac{100*101}{2}=50*101=5050$ (gauss reference – legend says that guy figured it out instantly with a similar argument when the teacher expected him to take a long time).

A sequence of the type where each term is some number more than the previous one (like the 1, 5, 9, 13, ... example) is called an arithmetic sequence. There is a formula for the sum of the first n terms in an arithmetic sequence, but first we need some notation.

I can write, for example

$$\sum_{n=1}^{100} 4n-3$$

The symbol on the left means "Sum", n=1 100 on the bottom and top means "Where n ranges from 1 to 100", so what we need to do is add 4n-3 as n goes from 1 to 100. Ie, we have to calculate:

$$4(1)-3+4(2)-3+4(3)-3+\cdots+4(99)-3+4(100)-3$$

Or

$$1+5+9+13+\cdots+393+397$$

If we start with a sum like the one above, we need to find an expression for the n'th term as discussed earlier, then determine the number of terms, in this case the last term is 397 and the expression is (from earlier) 4n-3, so the last term is the n'th term where 397=4n-3, so by simple algebra n=100.

We can split the sum as follows

$$\sum_{n=1}^{100} 4n-3=\sum_{n=1}^{100} 4n-\sum_{n=1}^{100} 3=4\sum_{n=1}^{100} n-\sum_{n=1}^{100} 3$$

Now the sum on the right is just adding 3 100 times as it is independent of n, so it is just 100. And the sum of the left is 5050 from earlier, or $\frac{100^2+100}{2}$ by that formula we derived. So,

$$1 + 5 + 9 + 13 + \cdots + 393 + 397 = \sum_{n=1}^{100} 4n - 3 = 4 * 5050 - 300 = 19900$$

Note that the variable we are summing over, in this case n, will never appear in the final expression for the sum: It is a "dummy" variable that we introduce just for this purpose. What I mean by this will become more clear by noticing the following:

$$\sum_{n=1}^{100} 4n - 3 = \sum_{r=1}^{100} 4r - 3 = \sum_{k=1}^{100} 4k - 3 = \sum_{skibiditoilet=1}^{100} 4skibiditoilet - 3$$

The point is, the index I am summing over should not come up in the final answer because it does not matter what I name it, as the expressions above all mean the same thing.

In general,

$$\sum_{n=1}^{k} An + B = A\sum_{n=1}^{k} n + \sum_{n=1}^{k} B = A\left(\frac{k^2+k}{2}\right) + Bk$$

Where $\frac{k^2+k}{2}$ comes from the formula we derived earlier.

So, $\sum_{n=1}^{k} An + B = \frac{A}{2}k^2 + \left(\frac{A}{2} + B\right)k = k\left(\frac{A(k+1)}{2} + B\right)$. Note that the first term (a) is A+B and the common difference between terms (d) is A, so the formula can be rewritten as $\frac{n}{2}[2a + (n-1)d]$ by rearranging where n is the number of terms, d is the difference between the terms, and a is the first term.

Also, we can define factorials as follows (by putting an exclamation mark after a number, or saying "number factorial": When someone does it with an exclamation mark you will never see it the same after this):

- $1! = 1$
- $2! = 1 * 2 = 2$
- $3! = 1 * 2 * 3 = 6$
- $4! = 1 * 2 * 3 * 4 = 24$
- $5! = 1 * 2 * 3 * 4 * 5 = 120$

And so on.

Now lets suppose we have a triangle with sides a, b, c and angles A, B, C opposite those sides respectively

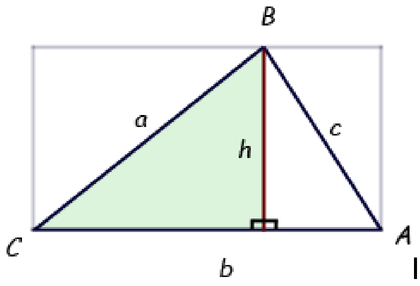


Image of such a triangle

Then the area is given by $\frac{1}{2}bh$ from earlier, but notice: $\sin(C) = \frac{h}{a}$ so $h = \text{a}\sin(C)$, so the area of the triangle is $\frac{1}{2}absin(C)$. So, if we know an angle and the two side lengths adjacent to that angle, we can work out the area of the triangle.

Note that this argument will work even if the h line coming from B does not end up on the line segment AC.

Now notice that if we rotate the triangle and apply the same argument, we will get that the area is equal to $\frac{1}{2}casin(B)$ and $\frac{1}{2}bcsin(A)$. Now these are all equal to the area of the triangle, hence equal to eachother, hence by doing some algebra there is a little rule we can derive:

$$\frac{1}{2}absin(C) = \frac{1}{2}casin(B)$$

$$\frac{1}{2}bsin(C) = \frac{1}{2}csin(B)$$

Since a is not 0, then

$$bsin(C) = csin(B)$$

$$\frac{sin(C)}{c} = \frac{sin(B)}{b}$$

But by exactly the same argument with a and b instead of c and b, we get the following triple equality:

$$\frac{sin(C)}{c} = \frac{sin(B)}{b} = \frac{\sin(A)}{a}$$

This is called the sine rule. It is important because if we know any angle and the length of the side opposite it, then once we have another angle or side we can work out its corresponding angle or side by solving for it using this rule. However, note that knowing the sin of an angle does not uniquely determine it, for example $\frac{1}{2} = \sin(30°) = \sin(150°)$. This is for 2 reasons:

1. The graph of $y = \sin(x)$ is symmetric about the line $y = 90°$
2. If we walk 30 degrees around a circle we will have the same y coordinate as when we have 30 degrees left to reach 180 degrees around the circle. This is actually the reason that the reason above is true.

Note that for sin, cos and tan we sometimes write $\sin(\theta)$ instead of $\sin(x)$ because θ represents angles.

Therefore when applying the sine rule, we may have enough information to narrow down two possible triangles but not any further, and extra information may be given in the context of the question to allow unique determination of the triangle.

Note that if x goes from 0 to 180 degrees then $\cos(x)$ goes from -1 to 1 so konwing the cosine does uniquely determine the angle. This is good because it means that for the cosine rule I am about to introduce, we do not have the ambiguous case.

The statement of the cosine rule is that if we are in the setup above, then

$$c^2 = b^2 + a^2 - 2abcos(C)$$

Note that if C is 90 degrees this is just pythagoras. Now here is a derivation of this rule from wikipedia:

The altitude through vertex C is a segment perpendicular to side c. The distance from the foot of the altitude to vertex A plus the distance from the foot of the altitude to vertex B is equal to the length of side c (see Fig. 5). Each of these distances can be written as one of the other sides multiplied by the cosine of the adjacent angle,[13]

$$c = a\cos\beta + b\cos\alpha.$$

(This is still true if α or β is obtuse, in which case the perpendicular falls outside the triangle.) Multiplying both sides by c yields

$$c^2 = ac\cos\beta + bc\cos\alpha.$$

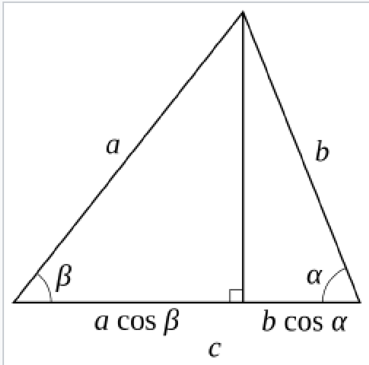


Fig. 5 – An acute triangle with perpendicular

The same steps work just as well when treating either of the other sides as the base of the triangle:

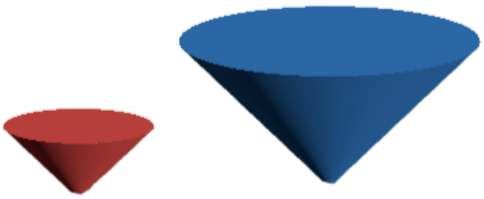
$$a^2 = ac\cos\beta + ab\cos\gamma,$$
$$b^2 = bc\cos\alpha + ab\cos\gamma.$$

Taking the equation for c^2 and subtracting the equations for b^2 and a^2,

$$c^2 - a^2 - b^2 = \cancel{ac\cos\beta} + \cancel{bc\cos\alpha} - \cancel{ac\cos\beta} - \cancel{bc\cos\alpha} - 2ab\cos\gamma$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

This proof includes an image to explain why it works using basic geometric identities.

Now suppose we have a shape and then we scale it by a factor of X. Below is an image to show exactly what I am talking about:



This image shows two cone shaped objects in which one has been scaled to be twice as long, wide and tall as the other. Now how much larger is its volume?

Well, we could first stretch it by 2x in the left-right direction, this multiplies the volume by 2, then we could do that in the front-back direction, now the volume is 4x bigger, then we could do that in the up-down direction, now the volume is 8x bigger. By this principle, if we scale something by a factor of X and we have n dimensions, the area/volume is multiplied by X^n. Also, surface area is like a dimension 2 object so it is multiplied by X^2. However, defining surface area rigorously and proving this for all cases is something very difficult that we will do in level 8 vector calculus. Therefore, we will show this for only a few shapes (For shapes with flat faces its immediate, for spheres and cones it comes from the upcoming formulae. On school tests you can usually assume you are being asked about one of these shapes or combinations of them)

Now, informally, the surface area of a shape is the area of its exterior, or how much paint you would need to cover it. What this means is that my cat has an unfathomably large surface area because she is very soft. Anyway, we can calculate the surface area of some simple shapes:

This cuboid shown has height 2, depth 3 and length 5. The surface area of this is the sum of the areas of all its faces: $2 * 3 + 2 * 3 + 2 * 5 + 2 * 5 + 3 * 5 + 3 * 5 = 62$.

Now there are some formulas for surface areas and volumes of shapes – the proofs for these are all deferred to level 4.

Volume of a sphere of radius r: $\frac{4}{3}\pi r^3$ (we expect that it would be proportional to r^3 by earlier discussion)

Surface area of a sphere of radius r: $4\pi r^2$ (hence scaling it scales the surface area by the square of the scale factor)

Volume of a cone with height h and base circle having radius r: $\frac{1}{3}\pi r^2 h$

Surface area of a cone with height h and base circle having radius r: πr^2 for the base circle (from the area of a circle) and πrl for the curved part. You can see that as r and l will scale by the same scale factor we scale a cone by, the surface area will scale by the square of that factor.

Now something that I will actually prove here: Note that the volume of a cylinder is the area of the base times the height, and the surface area of a cylinder is the circumference of the base times the height, ie we get the following:

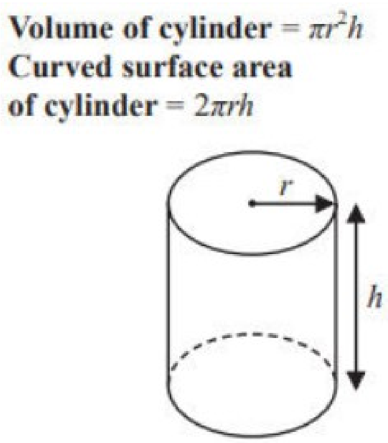


Image to show volume and curved surface area of cylinder

Note that the surface area of the flat parts is just $2\pi r^2$.

Note that volume and surface area have fundamentally different units – We are taking about units of area (squares) and units of volume (cubes).

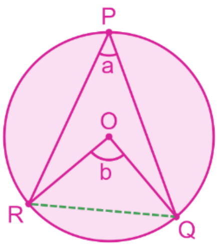
These formulas can be used to

i) Calculuate the surface area of a shape

ii) Calculate the radius or height of a shape given the surface area and other relevant information.

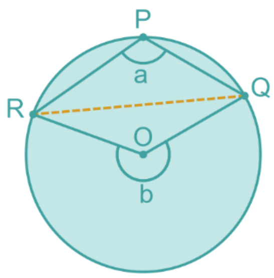
We also need some properties of circles that can be used to solve for lengths or angles.

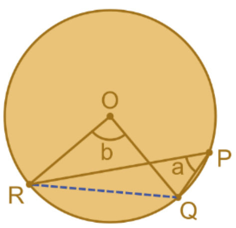
Note that if I have 3 points in 2D space (which from now on I will refer to as a plane, higher dimensional space is called a hyperplane) and they do not lie on the same line, it is easy to see that there is a circle whose circumference goes through those points. We will expand on this idea later.



This image shows a circle with center O. A very imporant theorem (which we will prove, then use it to derive the rest of our circle properties) says that in a configuration like this, the angle b is double the angle a.

This theorem also works even if the configuration looks like one of the ones shown below:





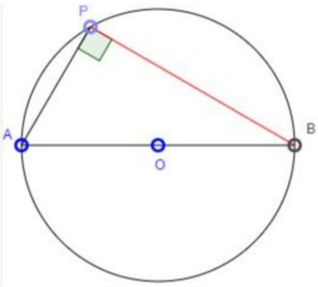
So lets prove this. In order to do this we need a way of naming angles. Note that in these diagrams the angle a is the angle between the line RP and the line PQ, and P is the point that the angle is at. We write the angle as RPQ. The point the angle is at is the second letter, and the first and third letter are other points on the lines that the angle is between.

Now in the pink circle in the image above, draw a line from O to R. Then OP, OR and OQ are all equal to the radius of the circle, so their lengths are equal to eachother. Therefore the triangles OPR and OPQ have two sides of the same length. This type of triangle is called an isoceles triangle. Note that it is now easy to see that the triangle OPR is symmetrical in the sense that it will not change if we reflect it about the line through O perpendicular to PR, and thus the angles ORP and OPR are the same. This is a general property of isoceles triangles – two of their angles are the same. Therefore, since the angles of the triangle OPR (meaning the triangle with vertices O, P and R) must add up to 180 degrees, it means that $2OPR + POR = 180°$. By the same argument, $2OPQ + POQ = 180°$. We can add these equations together to get $2OPR + 2OPQ + POR + POQ = 360°$. But we know that the angles b, POQ and POR form a circle and thus $b + POQ + POR = 360°$. But we also know that $OPR + OPQ = a$, and thus we have that $2a + 360° - b = 360°$ by putting all the last 3 equations we got together. Simplifying this gives $2a = b$ as required, and the same argument works for the blue circle in the image above. Thus we just need to deal with the orange circle in the image above.

For that, note that $OPQ - OPR = a$, so we want to show that $2OPQ - 2OPR = b$. Lets call E the point wherethe line segments RP and OQ intersect, then a+OQP+PEQ is 180 degrees since they are the angles of a triangle. It is easy to see from the diagram that the angles PEQ and OER are equal. Also since $a + OQP + PEQ = 180°$, $PEQ = 180° - a - QOP$ so we know that $OER = 180° - a - QOP$. But we know that $b + OER + ORP = 180°$ since those are the angles in the triangle OER. Hence, by the equation for OER, $b + 180° - a - QOP + ORP = 180°$, but from the fact that $OPQ - OPR = a$, we know that $OPQ - ORP = a$ as OPR=ORP by symmetry of the isoceles triangle ORP – same argument as above. Hence, $b + 180° - a - QOP + ORP = 180°$ means $b + 180° - 2a = 180°$ and therefore finally we get $b = 2a$.

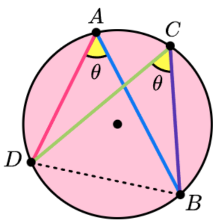
What we have done above is called angle chasing.

Now, as promised, from this theorem, we will see that the rest of the properties we will show come out easily and we have done the hard work.



This image is a reference image for the next circle property. If we have a triangle formed by three points that determine a circle and it just so happens that one of the sides of the triangle is a diameter, then the triangle is right angled. This is because from the image above and the theorem above, we know that $180° = AOB = 2APB$ since angles on a straight line are 180 degrees, hence $APB = 90°$.

Note also that if we have a right angled triangle and we put a circle that goes through its three points, then its hypotenuse (long side) will be the diameter (ie, chord through the center) of the circle. This is because (using the same point names as in the reference image) if AB was not the diameter, then the angle AOB would not be 180 degrees so the angle APB would not be a right angle.



This reference image shows another circle property, which is that angles coming from the same segment are equal. This is because if we call the center O, then by the first theorem, the angle DOB is both twice DAB and DCB using the point names from the reference image, hence DAB and DCB are equal to eachother.

Note that if we have two intersecting chords of a circle, as in the image below

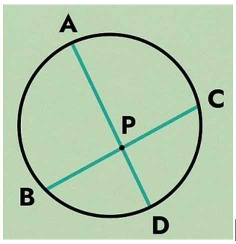
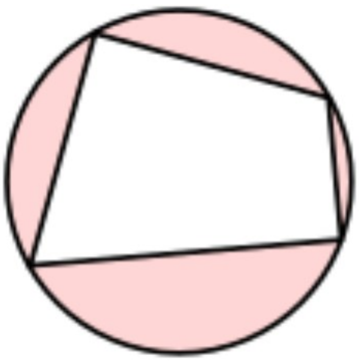


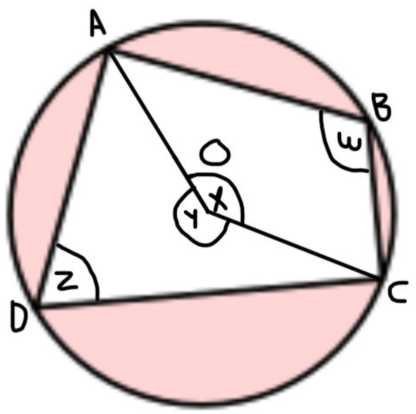
Image of a circle with 2 intersecting chords

Then we can show that (AP)(DP)=(BP)(CP) where by AP I mean the distance from A to P, and similarly for the rest of the distances. The proof is that if we make APB and PCD into triangles then notice that by the previous theorem, ABC and ADC are the same, and from the diagram APB and CPD are the same, and therefore CPD and APB are similar triangles since they share 2 angles – ie you can get from one to the other by reflecting and scaling it. Therefore, the scale factor between the two triangles is exactly $\frac{PD}{PB}$, but also $\frac{PA}{PC}$, therefore $\frac{PD}{PB} = \frac{PC}{PA}$. Multiplying both sides by (BP)(AB) gives exactly the desired result.

Now we will return to the “A circle is determined by 3 points” idea and talk about when it is determined by 4 points. So suppose we have 4 points that determine a circle. Now draw a quadrilateral with those 4 points as vertices, as shown in the image below:



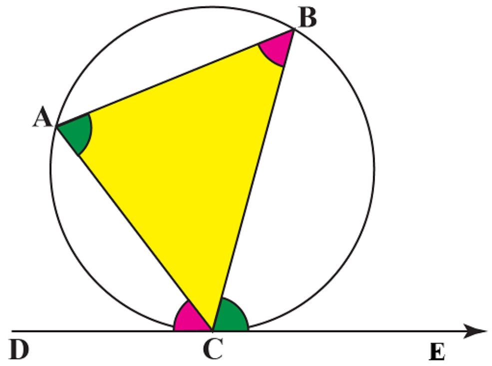
A quadrilateral whose vertices determine a circle is called a cyclic quadrilateral. We know its angles add up to 360 degrees but we can say something stronger: Suppose the vertices are A, B, C, D in clockwise order and the center is O. Then I will name some angles in a sketch below:



From the image, x+y=360, from the theorem above, x=2z and y=2w, hence z+w=180. For this reason, any two opposite angles in a cyclic quadrilateral add up to 180 degrees.

Now I want to show that any quadrilateral with this property (that each opposite angle pair adds up to 180 degrees) is cyclic. To do this, consider taking a quadrilateral with this property and putting a circle through three of the points A, B, C which we know we can do for sure. Then we want to show that D is on the circle. Let D’ be the point where AD intersects the circle (Of course, D=D’ but we don’t know this as we need to prove it). Then ABCD’ is a cyclic quadrilateral, but then $ABC + AD'C = 180°$, and since we assume $ABC + ADC = 180°$ it means $ADC = AD'C$. But then D=D’ as required, as because $ADC = AD'C$ and A, D and D’ are parallel, it is easy to see geometrically that D’C is parallel to DC if the angles satisfy $ADC = AD'C$, but if $D \neq D'$ this only happens if D' lies on the line DC, but since it also lies on the line AD by assumption, D’ is the unique intersection of the lines AD and DC which is D.

We have one last circle theorem called the alternate segment theorem, which says that if we draw a line tangent to a circle then mark angles as in the image below, then CAB=BCE and ABC=ACD.



Here is the reference image for this.

We will just prove that ACD=ABC as the other equality follows from the fact that ACD+ACB+BCE is 180 degrees and so is ABC+ACB+BAC as those are the angles of a triangle, and thus we know that in fact ACD+ACB+BCE=ABC+ACB+BAC and thus ACD+BCE=ABC+BAC, so if we can prove ABC=ACD then it will follow that also BAC=BCE.

By a theorem above, we can move B around on the circle and the and the angle ABC will not change. So lets make the convenient choice to move B such that BC is perpendicular to the line DE. If the line AC were in the way, we would simply move that one instead and do the same argument. Now once we have done that, BAC is a right angle as BC is the diameter of the circle and we can apply the relevant theorem that we proved earlier. But DCB will also be a right angle. We have$ABC + BAC + ACB = 180°$ so $ABC + ACB = 90°$, but also $ACB + ACD = 90°$, so finally we get that $ABC + ACB = ACB + ACD$ and hence $ABC = ACD$ which is what we wanted to show. This is called the alternate segment theorem.

Those are all the circle properties we cover in this level.

You may or may not have realized, or know, that if you take any fraction and write out its decimal expansion, ie write out the number, it will settle into a perfect repeating pattern. Lets prove this. Although this proof is not required for GCSE maths or equivalent, we provide it here.

We will prove this by demonstrating why this is always true using an example. The reason we talk about all this is because given a repeating decimal, we want to be able to convert between recurring decimals and fractions and it is very important that we know that this is always possible.

Consider $\frac{1}{7} = 0.142857142857142857 \ldots$. Then $\frac{10}{7} = 1.42857142857142857 \ldots$. This does not start with a 0 so what we do is we subtract 1 until it does, and each time we do this we need to subtract 7 from the numerator to keep the equation balanced. We get $\frac{3}{7} = 0.42857142857142857 \ldots$. We can repeat this procedure again: $\frac{30}{7} = 4.2857142857142857 \ldots$ so $\frac{2}{7} = 0.2857142857142857 \ldots$. Each time, the numerator will be either 0, 1, 2, 3, 4, 5, or 6. At some point, it will go back to where it started, and this will mean it must go into an endless loop as each numerator is uniquely determined by the previous one and only the previous one, and hence the decimal places will also go into an endless loop.

Also, even if the numerator did not start at 1, I could have subtracted 7 until it was strictly less than 7 and applied the same argument. I could have also applied the same argument if the denominator was any other non-zero integer to begin with.

Note that if a number's decimals do not repeat, it therefore means the number is **irrational:** it is not a fraction of integers. 1.01001000100001... is irrational as it does not technically repeat. So is pi, which we prove in the misc results section of this website, hence pi's decemals do not start to infinitely repeat.

We know that long division lets us convert a fraction into a recurring decimal, but we can do the opposite. To do this, we will – much to the dismay of u/SouthPark_Piano if you happen to get this reference – use the fact that 0.999999999...=1, as we proved in level 1, or alternatively because they would have to occupy the same position on the number line.

Because of this fact, we see that, eg $0.22222222222\ldots=\frac{2}{9}$. Also, $0.259259259259\ldots=\frac{259}{999}$. However, it just so happens that $\frac{259}{999}$ can be simplified to $\frac{7}{27}$. Also, even if a decimal number does not start with 0 or only eventually repeats, we can still do this. Here is an example that addresses both:

$$4.4781818181\ldots=4+0.47+0.0081818181\ldots=4+0.47+\frac{1}{100}*0.8181818181\ldots$$
$$=4+\frac{47}{100}+\frac{1}{100}*\frac{81}{99}$$

Now we just add and multiply and simply the fractions as we should know how to do and we get that the correct answer is $\frac{2463}{550}$.

From this procedure it is easy to see the converse of the theorem above – that any eventually repeating decimal number can be written as a fraction.

It is possible to add, subtract, multiply and divide and simplify algebraic fractions the same way you do with normal fractions. Here are some quick examples of this:

$\frac{x^2-5x+6}{x^2-9}$ can be simplified. To do this we will factorize the numerator (top) and the denominator (bottom). If we do that we get $\frac{(x-2)(x-3)}{(x+3)(x-3)}$. Therefore, for all x not equal to 3, this is just $\frac{x-2}{x+3}$.

It is useful to know the difference of squares rule: If you have the difference of two squares you can factor it. Ie, by simple algebra you can verify that $a^2-b^2=(a+b)(a-b)$ for any a and b. Therefore in an expression like $4x^2-9$ you can factor it as follows: Spot that $4x^2$ is the square of $2x$ (Not the square of 4x, as $(4x)^2\neq 4(x^2)$) and 9 is the square of 3, so we get $4x^2-9=(2x-3)(2x+3)$ by applying this rule.

Example: Lets solve the intimidating looking equation $\frac{x+6}{x+1}-\frac{1}{2x-4}=4$ (x is not -1 or 2 so we are not dividing by 0). As with fractions that just involve numbers, we want to get a common denominator. We can do this by first multiplying everything by x+1, then multiplying everything by 2x-4.

Multiplying by x+1: $x+6-\frac{x+1}{2x-4}=4(x+1)$

Multiplying by 2x-4: $(x+6)(2x-4)-(x+1)=4(x+1)(2x-4)$

Expanding: $2x^2 + 7x - 25 = 8x^2 - 8x - 16$. Moving everything to one side gives $6x^2 - 15x + 9 = 0$. Everything there is a multiple of 3 so we can simplify to $2x^2 - 5x + 3 = 0$. Solving this by either the quadratic formula, or spotting the factorization $(2x-3)(x-1)$, gives the solution: $x = 1$ or $x = \frac{3}{2}$.

We can solve simaltaneous equations that involve quadratics.

Example: $y = 11x - 2$ and $y = 5x^2$. Geometrically, we want to find where a line and parabola intersect, and we want to find x and y. From these equations, $5x^2 = 11x - 2$. To apply the quadratic formula we want to get 0 on the right hand side so we must move everything over: $5x^2 - 11x + 2 = 0$. The solutions in x to this are $x = \frac{1}{5}, x = 2$. For each of these values, we can use $y = 11x - 2$ to get that for $x = \frac{1}{5}, y = \frac{1}{5}$ and for $x = 2, y = 20$.

Example: $y = 2x - 11$ and $x^2 + y^2 = 25$. Luckily we know what y is in terms of x from the first equation so we can put it directly into the second equation: $x^2 + (2x - 11)^2 = 25$ which we can expand to get $5x^2 - 44x + 121 = 25$ so $5x^2 - 44x + 96 = 0$. I figure I should actually give an example of how the quadratic formula works in practice, so here is that example:

a=5, b=-44, c=96, so

$x = \frac{-(-44) \pm \sqrt{(-44)^2 - 4(96)(5)}}{2*5} = \frac{44 \pm \sqrt{16}}{10} = \frac{44 \pm 4}{10}$ which is 4 or $\frac{48}{10} = \frac{24}{5}$.

For each of these values of x, we can use $y = 2x - 11$ to work out y.

We will now introduce another concept which seems pointless but it is here because we will work more with in later levels.

A vector is basically an arrow in space.

For the purposes of this level you just need to know that a vector can be thought of as an arrow in 2 or 3 or higher dimensional space that starts from the origin or the point (0,0) and ends at some other point.

We typically write vectors as columns with each entry as the coordinate, or component. Precisely what I mean is, for example, if a vector in 2D space goes from (0,0) to (2,1) then we write it as $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

We can add and subtract them by adding and subtracting each of the components and we can multiply them by a fixed number. Adding two vectors is basically putting the tail of one at the tip of another and finding where the tip of the second one is.

Example: In 2D the equation of a vector that goes from (2,1) to (5,-2) would be the vector that you need to add to $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to get to $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ which is $\begin{pmatrix} 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$. If A is the point (2,1) and B is the point (5,2) then we would write this as $\overrightarrow{AB}$.

Also, if we have a translation where we move an object in space, such as a graph, we can use a vector to describe what we move it by.

Vectors have length in the usual sense.

Example: $\begin{pmatrix} 9 \\ 24 \\ 32 \end{pmatrix}$ by pythagoras has length $\sqrt{9^2 + 24^2 + 32^2} = 41$

In theory you can solve geometrical problems using vectors.

Example: To find the slope of the line in the image below which is assumed to be to scale, we note that in order for the line to go up 2 units it must go forward 3 units, and the slope is “rise over run” so we get that the slope is $\frac{2}{3}$

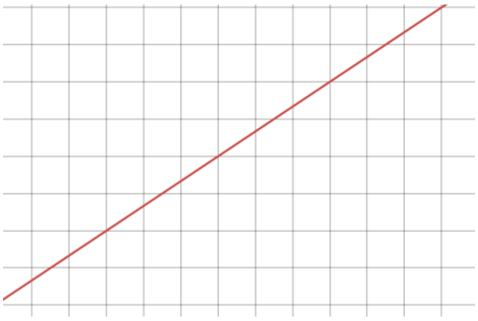


Image: a graph of a red line

Now try to think about the line that would make an angle of 90 degrees with this line and try to guess what the slope will be before reading on.

I will now add a perpendicular line to the plot.

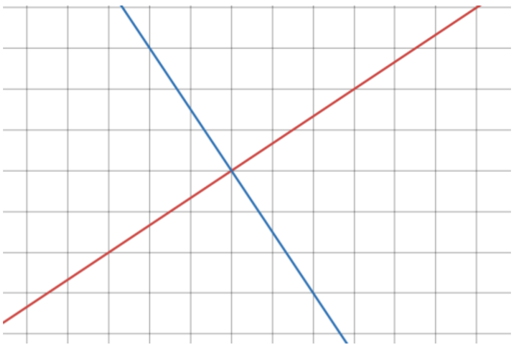


Image: The red line above and a line perpendicular to it.

Do you see it? The slope of the blue line is $-\frac{3}{2}$ by the same reason as above.

The general principle which holds for precisely the same geometric reason is the following key fact:

If a line has a slope of m, then the line perpendicular to that line has a slope of $-\frac{1}{m}$.