Contents

1 Methods of proof

There is a method you can use to prove things called "Proof by contradiction" where you essentially suppose that the thing you want to prove is false and then prove that it leads to an absurd conclusion.

I give two classic examples.

Theorem. There are infinitely many prime numbers

Proof. Suppose not, then we could write a list of prime numbers as $p_1, p_2, p_3, \dots, p_n$ and this is all of them. We know that we can factor any integer into prime numbers, so this must in particular include the integer $p_1 * p_2 * p_3 * \cdots * p_n + 1$. Then this is 1 more than a multiple of p_1 so not divisible by p_1, but this is also 1 more than a multiple of p_2 so not divisible by p_2, and so on for all of them. So this number is not divisible by any prime number. This is absurd, so the only way this is possible is if the premise (that there are finitely many prime numbers) is false.

□

Proposition. $\sqrt{2}$ is irrational, meaning we cannot write it as a fraction of integers $\frac{a}{b}$.

Proof. Suppose we could, then we could rearrange as follows:

$$\sqrt{2} = \frac{a}{b}$$

$$\sqrt{2}^2 = \left(\frac{a}{b}\right)^2$$

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

Now in the prime factorization of b^2, 2 must appear an even number of times, since it is a square number so 2 appears twice as many times as it does in b. But then by the same logic a^2 has an even number of 2's in its prime factorization. But this is absurd since $2b^2 = a^2$ so a^2 has 1 more 2 in its prime factorization than b^2 so they cannot be both even, so we have our contradiction.

□

We can also prove things by exhaustion, ie checking many cases.

Example. No square number ends in a 7.

Proof. Suppose we start with a number that ends in a 0, then its square ends in a 0.

□

If we start with a number that ends in a 1, then it can be written as 10k+1, so we get $(10k+1)^2 = 10(10k+2)+1$ which ends in a 1. We can do this for the other 8 possible end digits and we will arrive at our conclusion.

Example. It is not true that adding two irrational numbers gives an irrational number

Proof. We do "proof by counterexample". If we can find one situation in which something is false, we know it is not always true.

So my example is we have the irrational numbers $\sqrt{2}$ and $2-\sqrt{2}$. Then their sum is 2 which is rational.

□

2 Miscellaneous tricks

Quick example: We know what the graphs $y = x$ and $y = \frac{1}{x}$ look like. Now try to think about what the graph of $y = x + \frac{1}{x}$ might look like if we add every y-coordinate in the two graphs together.

In the end, figure 1 is our answer, hopefully it makes sense why.

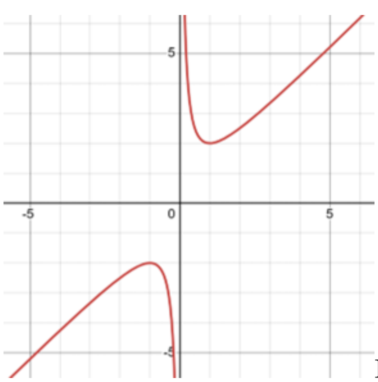

Figure 1

Also note that from level 3.1 we basically have the tools to differentiate anything we want but integration is generally harder. It is for this reason that in coming levels we will learn tricks for integration.

Another thing which is generally hard is to solve polynomials of degree greater than 2, where typically if we cannot spot integer roots, numerical methods are more practical. It is for this reason that we will also learn some numerical methods.

One trick which can be useful sometimes is if we have a bunch of things equal to 0, this is the same as their product being 0. I will give an example of this.

If we want to make a single equation that will output the graph of the lines y=x, y=2x, then this is the same as saying either y-x is 0 or y-2x=0. So we want the graph

$(y-x)(y-2x)=0$ which is the same as $y^2-3xy+2x^2=0$ which we can actually factor if we are clever about it, it is a pseudo-quadratic if we write it as $x^2\left(\left(\frac{y}{x}\right)^2-3\left(\frac{y}{x}\right)+2\right)=0$. However, this is misleading as it suggests $x^2=0$, but the reason this does not work is because x^2 is not being multiplied by something defined whenever it is 0, but this is just a technicality and this example is not that important it's just cute.

On the topic of factorizations, here are some useful ones to know.

We can check directly that

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

$$a^4-b^4=(a-b)\left(a^3+a^2b+ab^2+b^3\right)$$

And the pattern continues, we have

$$a^n-b^n=(a-b)\left(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+b^{n-1}\right)$$

We also have

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)$$

And similarly the pattern continues for larger odd powers – this one only works for odd powers.

3 Trigonometry

3.1 The R-method

Example. Suppose we want to simplify $A*\sin(x)+B*\cos(x)$. The method to do this is as follows:

Write $A*\sin(x)+B*\cos(x)=\sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}*\sin(x)+\frac{B}{\sqrt{A^2+B^2}}*\cos(x)\right)$. We do this because now if we take $\frac{A}{\sqrt{A^2+B^2}}$ and $\frac{B}{\sqrt{A^2+B^2}}$ and add their squares we get 1. Therefore, it is possible to write $\frac{A}{\sqrt{A^2+B^2}}=\cos(t)$ and $\frac{B}{\sqrt{A^2+B^2}}=\sin(t)$ where $t=\arccos\left(\frac{A}{\sqrt{A^2+B^2}}\right)$. We now have

$$\sqrt{A^2+B^2}\left(\cos(t)*\sin(x)+\sin(t)*\cos(x)\right)$$

But this is the familiar formula that simplifies to $\sqrt{A^2+B^2}\sin(x+t)$.

We can always do this and in particular it means that the graph of $A*\sin(x)+B*\cos(x)$ will look like a sine wave and the maximum and minimum values it attains will be at $\frac{\pm A}{\sqrt{A^2+B^2}}$.

3.2 Derivatives and integrals

We will now take arcsin and try to differentiate this. I will do this more carefully than most textbooks and explain where most textbooks go wrong.

Here we assume x is real and $|x|\leq 1$, as that is the domain that arcsin is typically defined.

$$y=\arcsin(x)$$

$$\sin(y)=x$$

Although $\sin(y) = x$ is implied by $y = \arcsin(x)$, the converse is not true, as discussed in previous levels.

$$\cos(y) = \frac{dx}{dy}$$

Here, many people assert that $\cos(y) = \sqrt{cos^2(y)} = \sqrt{1 - sin^2(y)}$, however the assertion $\cos(y) = \sqrt{cos^2(y)}$ is only true if cos(y) is positive. Luckily it is, since the range of arcsin is by definition $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ where cos is always positive, but usually this step is not done properly. The result for the derivative of arcsin follows, and the derivative of arccos can be derived in the same way, noting that sin is positive and -sin is negative in the range of arccos.

Now I will show a new trick for integration.

Suppose we want to find $\int \cos(7x+5)dx$. This is something we know how to integrate applied to something of the form Ax+B. Naively, we might say the integral is $\sin(7x+5)+c$, which is wrong but it is a good starting point. Lets try to differentiate it and see what happens. Recall the chain rule: We end up with $\cos(7x+5) * \frac{d(7x+5)}{dx} = 7\cos(7x+5)$. Now we know what to do: We just got to a constant multiple of the thing we were trying to integrate. Therefore, $\frac{1}{7}\sin(7x+5)+c$ is the correct antiderivative. This brings us to the next section

4 Integration

4.1 Reverse chain rule

As another example, lets try to integrate $\int \sin^2(x)dx$. It is not immediate how to do this until you recall the identity $\cos(2A) = 1 - 2\sin^2(A)$. We can now rearrange this to get $\sin^2(x) = \frac{1-\cos(2x)}{2}$, which is something we know how to integrate. The integral of $\frac{1}{2} - \frac{\cos(2x)}{2}$ using the same reverse chain rule trick is just $\frac{1}{2}x - \frac{\sin(2x)}{4} + c$.

Now I will use the chain rule to differentiate $\ln(f(x))$ for some function f that is differentiable. This will be useful as the result will be something we can use to calculate many integrals. To do this we will recall the result that the derivative of $ln(x)$ is $\frac{1}{x}$.

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)}\frac{d}{dx}f(x) = \frac{f'(x)}{f(x)}$$

Using this, we can find, say, $\int \frac{2x}{1+x^2}dx$ by spotting that the numerator is the derivative of the denominator. We get the answer, by setting $f(x) = 1 + x^2$, that the antiderivative is $\ln(1+x^2)+c$.

We will now find $\int \tan(x)dx$. Before I do this, I will find the derivative of $\tan(x)$ instead of its antiderivative, just because this is an easier and useful result. Recalling that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and using the quotient rule for differentiation we get $\frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$ as the derivative.

To find $\int \tan(x)dx$, note that we can rewrite this as $\int \frac{\sin(x)}{\cos(x)}dx$. This almost looks like the situation where the numerator is the derivative of the denominator, we just need a minus sign. So we will instead try to evaluate $-\int \frac{-\sin(x)}{\cos(x)}dx$. When the numerator is in fact the derivative of the denominator, the antiderivative is just ln of the denominator. We therefore get $\int \tan(x)dx = -\ln(cos(x)) + c$.

We will to do another derivative: lets try to find $\frac{d}{dx}\arctan(x)$, recalling that $\arctan(x)$ is defined by convention to take values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

So as usual, if $y = \arctan(x)$ then $x = \tan(y)$ so $\frac{dx}{dy} = \sec^2(y)$. But remember we have the identity that $\sec^2(y) = 1 + \tan^2(y)$ and so $\frac{dx}{dy} = 1 + \tan^2(y) = 1 + x^2$. Therefore, $\frac{dy}{dx} = \frac{1}{1+x^2}$.

This result is useful because now we know that $\int \frac{1}{1+x^2}dx = \arctan(x) + c$.

4.2 Integration by substitution

Remember how we integrated $\int \frac{2x}{1+x^2}dx$ using reverse chain rule? Because there is a more general method we can use called integration by substitution. I will show how to do it for this example.

Lets define u to be $1+x^2$. So the integral becomes $\int \frac{2x}{u}dx$. But we can change the dx into a du. What I will do right now is very dodgey, but in level 4 we will justify it – it is not difficult to justify by considering the derivative of the integral, or to get a feel for what is happening by thinking of dx and du as tiny changes in x and u that we sum over. But here is the dodgey thing: We write $du = \frac{du}{dx}dx$. It is dodgey because du and dx are not really well defined objects, at least we have not defined them other than just being notation conventions. In this case, $\frac{du}{dx} = 2x$ so we can replace $2xdx$ with du. So we end up with $\int \frac{1}{u}du$. Now this is something we can evaluate, it is just $\ln(u) + c$, which is $\ln\left(1+x^2\right) + c$.

It is like reversing the chain rule because the derivative of $f(u)$ is $\frac{du}{dx}\frac{df}{du}$.

If you aren't sure what substitution to make, often it is best to just guess. It is generally a good sign if you can easily pull out a factor of $\frac{du}{dx}$ as in the above example where there was an obvious factor of 2x.

Sometimes the substitution may not be obviously in the integral: For example I will use the substitution $u = \sin(x)$ to find $\int \cos^3(x)dx$. It is certainly not obvious how this works so I will walk through it. Write

$$\int \cos^3(x)dx = \int \cos(x)\left(\cos^2(x)\right)dx = \int \cos(x)\left(1-\sin^2(x)\right)dx$$

But note that $du = \cos(x)dx$ (When I say these are equal, I mean that you can replace one with the other in the integral). Therefore we get $\int 1-u^2du = u - \frac{u^3}{3} + c = \sin(x) - \frac{\sin^3(x)}{3} + c$.

If we are finding definite integrals, there is a rule on how to change the limits. I will demonstrate this using an example: Lets find $\int_0^1 \frac{1}{1+e^x}dx$ using the substitution $u = e^x$. I know that we do not see a factor of $e^x dx$ that becomes du, but it is the case that $\frac{du}{dx} = e^x = u$ so $dx = \frac{du}{u}$. Therefore we write $\frac{1}{1+e^x}dx$ as $\frac{1}{1+u}\frac{du}{u}$. But notice: Here, x is ranging from 0 to 1, so u must range from 1 to e. So we have to change the limits on the integral. We get $\int_1^e \frac{1}{u(1+u)}du$, which we know how to evaluate using partial fractions. If you do it correctly, you should get $1+\ln(2)-\ln(1+e)$ as your answer.

4.3 Integration of algebraic fractions

We now have the tools we need to integrate a general class of functions, specifically any function that is a polynomial divided by another polynomial of degree up to 2. Here is how we do this. It may be confusing but I will do some examples:

First, we do polynomial long division to get a resulting polynomial, so we reduce to the case where we have a remainder term. If it is of the form $\frac{A}{Bx+C}$ we can integrate that with the reverse chain rule. But in general we can look into how to integrate things like $\frac{Ax+B}{Cx^2+Dx+E}$.

To do that integral, we factorize the denominator using the quadratic formula and do it by partial fractions if this is possible. If not, it means that when we complete the square, we will end up with a denominator like $C(x+\lambda)^2+\mu$ where $\mu > 0$ so we cannot take the square root and get a real solution. In this case, we do the substitution $u = x+\lambda$ which will not change dx. We now have to integrate something like $\frac{Ax+B}{Cx^2+D}$. We can deal with the "Ax" term by a log reverse chain rule trick, leaving the term $\frac{B}{Cx^2+D}$. Lets simplify this by multiplying by constants and we will have a term like $\frac{1}{x^2+A}$. Since A is positive, write this as $\frac{1}{x^2+A^2}$. This is now very close to something we can integrate, as recall we derived earlier $\int \frac{1}{1+x^2}dx = \arctan(x) + c$, we just need to use the substitution $u = \frac{x}{A}$ to get that we want. We get $\int \frac{A}{A^2u^2+A^2}du = \frac{1}{A}\arctan(u) + c = \frac{1}{A}\arctan\left(\frac{x}{A}\right) + c$. Now, 2 concrete examples:

Example.

$$\int \frac{4x^2-5x+2}{x^2+x-2}dx$$

Long divison:

$$\int 4 + \frac{-9x+10}{x^2+x-2}dx$$

Note that $\frac{2x+1}{x^2+x-2}$ integrates to $\ln\left(x^2+x-2\right)$ since the derivative is the numerator of the denominator, and the 4 integrates to 4x. We now have

$$4x - \frac{9}{2}\ln\left(x^2+x-2\right) + \int \frac{14.5}{x^2+x-2}dx = 4x - \frac{9}{2}\ln\left(x^2+x-2\right) + \int \frac{14.5}{(x+2)(x-1)}dx$$

And we know how to do partial fraction shenanigans.

$$\int \frac{x^3+6x^2+15x+7}{x^2+6x+13}dx$$

Long division,

$$\int x+\frac{2x+7}{x^2+6x+13}dx=\frac{x^2}{2}+\int \frac{2x+7}{x^2+6x+13}dx$$

The derivative of the numerator is $2x+6$. So $\frac{2x+6}{x^2+6x+13}$ will integrate to $\ln\left(x^2+6x+13\right)$. We are thus left with

$$\frac{x^2}{2}+\ln\left(x^2+6x+13\right)+\int \frac{1}{x^2+6x+13}dx$$

Completing the square on the denominator,

$$\frac{x^2}{2}+\ln\left(x^2+6x+13\right)+\int \frac{1}{(x+3)^2+4}dx$$

But $4=2^2$ so the integral is

$$\int \frac{1}{(x+3)^2+2^2}dx$$

And we know how to integrate this: Since it does not matter that the x is + a constant, this is just

$$\frac{1}{2}\arctan\left(\frac{x+3}{2}\right)+c$$

So the final answer is

$$\frac{x^2}{2}+\ln\left(x^2+6x+13\right)+\frac{1}{2}\arctan\left(\frac{x+3}{2}\right)+c$$

Personally I can never remember the formula $\int \frac{1}{x^2+a^2}dx=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$, because I'm lazy and just rederive it on the spot when needed.

Now, another example of integration by substitution: Lets find

$$\int_1^{\sqrt{2}} x^3\left(x^4+1\right)^7dx$$

Of course, we see x^4+1, and a factor of x^3dx which is a quarter of its derivative, so the clear substitution to make is $u=x^4+1$ which will give $\int_{x=1}^{\sqrt{2}} x^3(u)^7dx=\frac{1}{4}\int_{x=1}^{\sqrt{2}}(u)^7du$

Always writing "x=" for the integration limits is a good habit because you remember whether you have changed them. We will change them now: $1^4+2=2$, $\sqrt{2}^4+1=5$. We now have

$$\frac{1}{4}\int_{u=2}^{5} u^7du=\frac{1}{4}\left[\frac{u^8}{8}\right]_2^5=\frac{1}{32}\left[5^8-2^8\right]=\frac{390369}{32}$$

Wow that's not a nice answer.

Now I want to point something out. If we have $\int \frac{1}{x^2-1}dx$ we could say that is $\int \frac{1}{x^2+i^2}dx$ where i is the square root of -1, and then this is just $\frac{1}{i}\arctan\left(\frac{x}{i}\right)+c$. In level 6 we will do a similar example under "a tricky integral problem" where we show that this is not technically wrong, but the convention is to always give your answer in a way that it involves real numbers.

4.4 Integration by parts

Now there is one more useful method for integration called integration by parts. It is essentially the reverse product rule.

The product rule says that $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$. Integrating both sides gives

$$uv=\int u\frac{dv}{dx}dx+\int v\frac{du}{dx}dx$$

But we typically write the integration by parts rule as

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Now I will do an example. Lets say we want to integrate $\int xe^x dx$. To do this, I will set x to be the thing to differentiate and e^x to be the thing to antidifferentiate. Ie,

$$u = x, \frac{dv}{dx} = e^x, v = e^x, \frac{du}{dx} = 1$$

Here we don't worry about +c, as we just need one antiderivative. We now write

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c = (x-1)\,e^x + c$$

Now another example. Lets find $\int \ln(x)dx$. It is not obvious how to do this, but it turns out it will work if you set $\frac{dv}{dx} = 1$ and $u = ln(x)$. Here we get $v = x$ and $\frac{du}{dx} = \frac{1}{x}$. We note that $v\frac{du}{dx} = 1$ which is certainly something we know how to integrate, so this looks promising. We have

$$\int \ln(x)dx = x\ln(x) - \int x * \frac{1}{x}dx$$

By the integration by parts formula. But $x * \frac{1}{x}$ is just 1 which integrates to x+c. Therefore,

$$\int \ln(x)dx = x\ln(x) - x + c$$

We can also find $\int \arctan(x)dx$ the same way. I won't go through all the details but I will sketch it out. We end up with $\int \arctan(x)dx = x\arctan(x) - \int \frac{x}{1+x^2}dx$

And we can do a reverse chain rule log trick. We can also do

$$\int \arcsin(x)dx = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx$$

And we can solve this with the substitution $u = \sqrt{1-x^2}$.

Now, another example, this time a definite integral:

$$\int_0^\pi x\sin(x)dx$$

In general if something is multiplied by x, it is a good idea to set x to be the part to differentiate. So we have $u = x, \frac{dv}{dx} = \sin(x),\ v = -\cos(x), \frac{du}{dx} = 1$. The integration by parts formula gives

$$\int x\sin(x)dx = -x\cos(x) - \int -\cos(x)dx$$

But remember, it is a definite integral. So what we do with the "uv" term is just evaluate it at π, and subtract the result when we evaluate it at 0. Ie, we have

$$\int_0^\pi x\sin(x)dx = [-x\cos(x)]_0^\pi + \int_0^\pi \cos(x)dx$$

Note that by a symmetry of the graph (shown in figure 2)

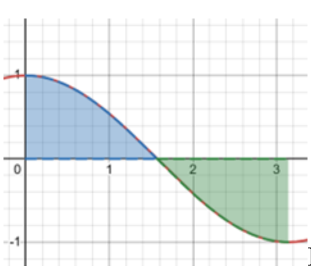

Figure 2

The integral on the right is 0. So we just have to get $-\pi\cos(\pi) - (-0\cos(0))$. The right hand term is 0, and $\cos(\pi)$ is -1, so our answer is just π.

Now, another example. Lets find

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \cos(x) dx$$

There is a trick to this which I will walk through. Set

$$u = e^x, v = \sin(x), \frac{du}{dx} = e^x, \frac{dv}{dx} = \cos(x)$$

Now

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \cos(x) dx = [e^x \sin(x)]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \sin(x) dx = e^{\frac{\pi}{2}} - \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \sin(x) dx$$

Which doesn't look like it has helped much, but now lets integrate by parts again. Consider $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \sin(x) dx$ and set $u = e^x, v = -\cos(x), \frac{du}{dx} = e^x, \frac{dv}{dx} = \sin(x)$

Being careful with our minus signs, $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \sin(x) dx = [-e^x \cos(x)]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} -e^x \cos(x) dx$

But we had $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \cos(x) dx = e^{\frac{\pi}{2}} - \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \sin(x) dx$, so we can substitute our above expression for $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \sin(x) dx$ to get

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \cos(x) dx = e^{\frac{\pi}{2}} - \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}} - [-e^x \cos(x)]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} -e^x \cos(x) dx$$

Now the trick is we have exactly the integral we want on the right hand side. So we can solve for it.

$$2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \cos(x) dx = e^{\frac{\pi}{2}} - \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}} + [e^x \cos(x)]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \cos(x) dx = e^{\frac{\pi}{2}} - \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}} + 0 - \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}}$$

So the final answer is

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^x \cos(x) dx = \frac{1}{2} e^{\frac{\pi}{2}} - \frac{\sqrt{2}}{2} e^{\frac{\pi}{4}}$$

Now, one more example: $\int_1^4 e^{\sqrt{x}} dx$. This does not look immediately like something we can solve, so we will guess that a substitution $u = \sqrt{x}$ might happen to work. Here, we have $du = \frac{1}{2\sqrt{x}} dx$ so $2udu = dx$. Indeed we get $\int_{x=1}^4 e^u dx = \int_{u=1}^2 2ue^u du$. We know how to integrate this – we did it before using integration by parts. We get $[2(u-1)e^u]_1^2 = 2e^2$.

5 Implicit differentaition

Another class of problems you can do is as follows.

Suppose y is a function of x which you can differentiate, and there is some equation relating y and x, as an example I will give the circle equation: $x^2 + y^2 = 1$. Then we want to find $\frac{dy}{dx}$ in terms of x and y. We do this by differentiating both sides. Here y is some function of x, and we can use the chain rule on it, since we assumed in the statement of the problem that y is differentiable, which just means you can differentiate it.

Here is how we differentiate both sides:

$$x^2 + y^2 = 1$$

$$\frac{d}{dx} x^2 + \frac{d}{dx} y^2 = \frac{d}{dx} 1$$

$$2x + \frac{dy}{dx} \frac{d}{dy} y^2 = 0$$

Since chain rule allows us to treat these derivatives as fractions.

$$2x + 2y\frac{dy}{dx} = 0$$

Therefore, $\frac{dy}{dx} = -\frac{x}{y}$.

As you can see, our assumption that y as a function of x is differentiable was reasonable in this case, since a circle is certainly differentiable. However, it is not obvious that this assumption is always valid, so we will discuss this further in level 6 (in the document about implicit differentiation). Unfortunately, the full justification is so complicated that I was unable to fit it into level 4 without making level 4 basically contain level 6 technical results.

Example. Suppose $5x^2 + 7y^3 - 2xy + \sin\left(e^{x-2y}\right) = 8$. Here we have to use the product and chain rules. For those curious I will show a picture of the graph (Figure 3).

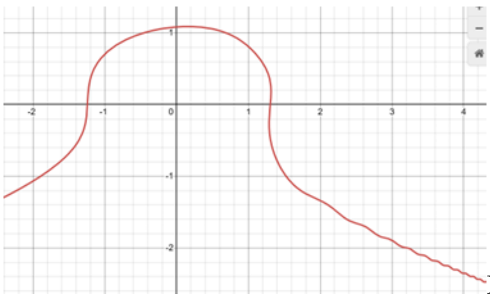

Figure 3

Surprisingly, despite the oscillations in the bottom right hand corner, we can find the slope in terms of x and y. Note that it will be something divided by 0 (hence undefined) at the vertical points.

Lets do the differentiation. You're gonna need to know your product and chain rules to follow this, because I will use them a lot. You will probably never be tested on an example this complicated in school, I just wanted to do it here.

$$10x + \frac{d}{dx}7y^3 - \frac{d}{dx}2xy + \frac{d}{dx}\sin\left(e^{x-2y}\right) = 0$$

$$10x + \frac{dy}{dx}\frac{d}{dy}7y^3 - \left[2x\frac{d}{dx}y + y\frac{d}{dx}2x\right] + \cos\left(e^{x-2y}\right)\frac{d}{dx}\left[e^{x-2y}\right] = 0$$

$$10x + 21y^2\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y + \cos\left(e^{x-2y}\right)e^{x-2y}\frac{d}{dx}\left[x - 2y\right] = 0$$

$$10x + 21y^2\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y + \cos\left(e^{x-2y}\right)e^{x-2y}\left[1 - 2\frac{dy}{dx}\right] = 0$$

$$10x + 21y^2\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y + \cos\left(e^{x-2y}\right)e^{x-2y} - 2\cos\left(e^{x-2y}\right)e^{x-2y}\frac{dy}{dx} = 0$$

$$10x + \left[21y^2 - 2x - 2\cos\left(e^{x-2y}\right)e^{x-2y}\right]\frac{dy}{dx} - 2y + \cos\left(e^{x-2y}\right)e^{x-2y} = 0$$

$$\frac{dy}{dx} = \frac{2y - 10x - \cos\left(e^{x-2y}\right)e^{x-2y}}{21y^2 - 2x - 2\cos\left(e^{x-2y}\right)e^{x-2y}}$$

So that's how that works.

6 Parametric equations

6.1 Introduction

Now we will talk about parametric equations. These basically allow you to draw graphs that are not just the graph of $y = f(x)$. As an example, if you draw a circle, your x-coordinate is given by $\cos(t)$ and your y-coordinate is given by $\sin(t)$, and if you plot the points $(\cos(t), \sin(t))$ letting t vary continuously with time, you will get a circle. This is how a parametric equation works. I will show some examples.

Figure 4 shows the graph of the example of $(\cos(6t), \sin(7t))$. Essentially your y-coordinate oscillates slightly faster than your x-coordinate, and this is the result. The convention is to use t for this purpose.

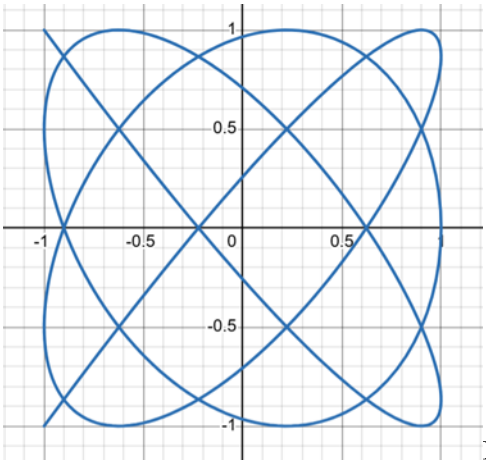

Figure 4

As another example, Figure 5 shows the parametric graph $(tcos(t),\ tsin(t))$ for $t > 0$.

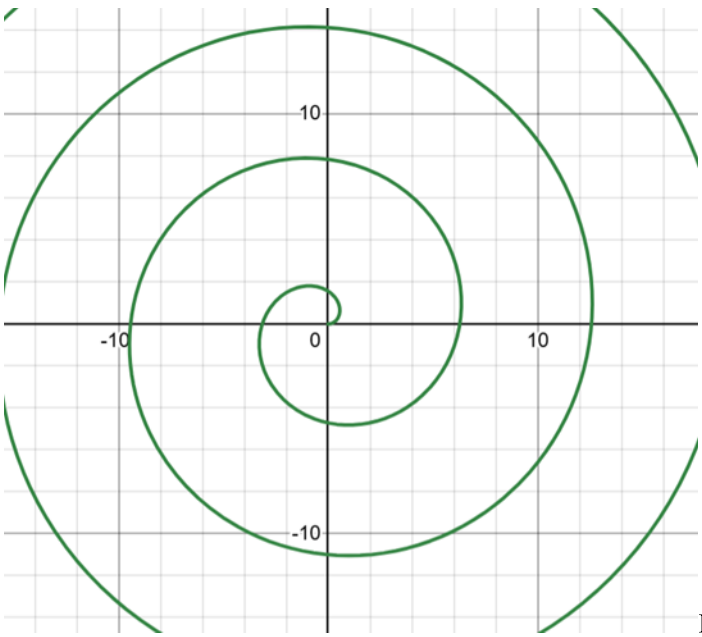

Figure 5

I could spend all day making cool graphs with this – you should do the same! But we need to do some actual maths. The most reliable way to sketch these curves is with a table of values and connecting points.

In some cases, it is possible to find y in terms of x by finding t in terms of x via rearranging, for example in the circle case of $(\cos(t), \sin(t))$ (where here we will constrain t to be between 0 and $\frac{\pi}{2}$ so that sin has a proper inverse function and so that we can safely take the square roots that we will take) we note that $\cos(t) = x$ so $t = \arccos(x)$ so $y = \sin(arccos(x)) = \sqrt{1 - \cos^2(\arccos(x))} = \sqrt{1 - x^2}$. This indeed agrees with the equation for a circle, as we observe that $y = \sqrt{1 - x^2}$ implies that $y^2 = 1 - x^2$ so $y^2 + x^2 = 1$. Often x in terms of t is a simple function we can invert.

6.2 Differentiation

Now we will take these parametric curves and find their derivative in terms of t. Here is how we will do that.

Lets take the example of the spiral: $(tcos(t),\ tsin(t))$. Here we know $x = tcos(t)$ and $y = tsin(t)$, so therefore $\frac{dx}{dt} = \cos(t) - t\sin(t)$, $\frac{dy}{dt} = \sin(t) + t\cos(t)$. We now observe:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin(t) + t\cos(t)}{\cos(t) - t\sin(t)}$$

As far as I know we can't get this in terms of x and y easily, but what we can do is notice that when the numerator is 0, we will have a horizontal stationary point (ie a flat point) and when the denominator is 0 and the derivative is "infinity" we will have a vertical stationary point. If the numerator and denominator are **both** 0 then there's not much we can do about that.

The reason we can do $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ is because we can multiply both sides by $\frac{dx}{dt}$ and use the chain rule.

As for why the curve generated by something parametric is something you can differentiate in the first place, it is because: We want to consider f defined by $y(t) = f(x(t))$, then f would be defined by $f(a) = y\left(x^{-1}(a)\right)$. If $\frac{dx}{dt}$ is not 0 near a, and $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both exist, then the inverse function of x is also something you can differentiate. We will formalize this much later, but just look at it: it is sufficiently convincing that this is true to say: Notice, if you can take the slope, then you can take the slope of the inverse by reflecting about the line $x = t$ (we will discuss this point in level 4), and if $\frac{dx}{dt}$ is not 0 near a then x is increasing or decreasing near a, so it does not hit the same point twice near a, ie is not many-to-one, so its inverse/reflection is not one-to-many, and hence is a genuine function. This also justifies our implicit claims from earlier about being able to differentiate stuff like log and arcsin.

6.3 Integration

We will also do integration with parametric curves. But we will be careful: Sometimes you are trying to find the area of a parametric curve, but if it loops back on itself like in the figure 6, you may double count some area. At least in A level, in all questions set, the curves do not do this.

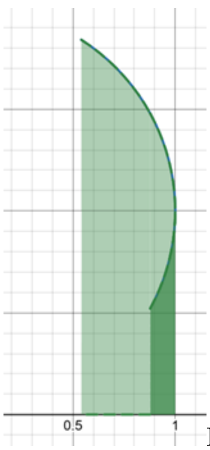

Figure 6

Lets now do an example. Suppose that $x = t^2, y = \frac{\sin(t)}{t}$, and t ranges from 0 to $\frac{\pi}{2}$. We get some curve, and we want to find the area under this curve. Here is how we do this: We have

$$\int_{t=0}^{\frac{\pi}{2}} y(t)\,dx = \int_{t=0}^{\frac{\pi}{2}} \frac{\sin(t)}{t}\,dx$$

Now we know how to do integration by substitution: We just need to substitute x for t to get the dx into a dt. We can do this by writing $x = t^2$ so $\frac{dx}{dt} = 2t$ so $dx = 2tdt$, since that is how you do integration by substitution. We get

$$\int_{t=0}^{\frac{\pi}{2}} y(t)\,dx = \int_{t=0}^{\frac{\pi}{2}} \frac{\sin(t)}{t}\,dx = \int_{t=0}^{\frac{\pi}{2}} \frac{\sin(t)}{t}\,2tdt = \int_{t=0}^{\frac{\pi}{2}} 2\sin(t)dt = 2$$

So that's our answer.

7 Volumes of revolution

7.1 How the method works

Now here is another cool type of integral we can do.

Imagine we have some function and we rotate its graph about the x axis, then we want to find the volume of the surface generated.

For example, lets say we have the line $y = x$, then if we rotate it, the surface generated will look like figure 7. We will aim to find the volume over $0 < x < 1$.

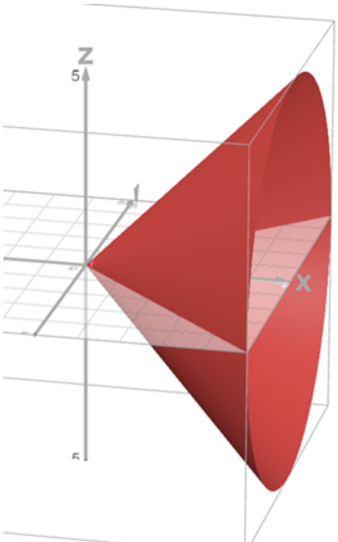

Figure 7

Here is how this will work: The formula is if the function is y, and we are looking at $a < x < b$, we just need to integrate $\int_a^b \pi y^2 dx$. Here is the reason why this is the formula:

Here you can see from the lazy quick sketch diagram below that the volume of revolution will approach as the width of these cylinders gets small the sum of the volumes of the cylinders which are each equal to $\pi y^2 dx$ and using the idea of integration as a sum from the separation of variables section we see that summing these volumes and taking a limit as dx 0 is the same as the integral typically used which is $\int \pi y^2 dx$.

Figure 8 shows a bad diagram of concentric thin cylinders to illustrate why the formula works.

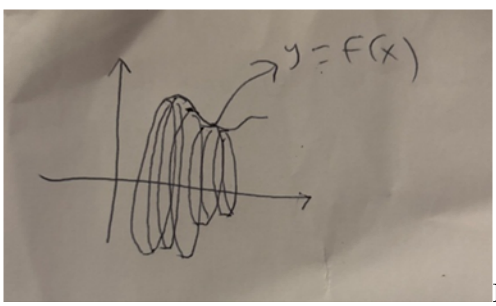

Figure 8

I will use this to derive some results from level 2.

7.2 Proving volume formulas

Proposition. The volume of a cone formula we stated in level 2 ($\frac{1}{3}\pi r^2 h$) is true

Proof. Intuitively, the volume should scale with the height times the square of the radius. The volume of a cone with radius r and height h is equal to the volume bounded by the line segment connecting (0, r) to (h, 0) after rotating it 2π radians about the x axis, as that volume is literally the cone on its side. The equation for this line segment is $y = r - \left(\frac{r}{h}\right) x$. We thus need to evaluate $\int_0^h \pi \left(r - \left(\frac{r}{h}\right) x\right)^2 dx$.

$$= \pi \int_0^h r^2 - 2r\left(\frac{r}{h}\right) x + \left(\frac{r}{h}\right)^2 x^2 dx.$$

$$= \pi r^2 \int_0^h 1 - \left(\frac{2}{h}\right) x + \left(\frac{1}{h^2}\right) x^2 dx.$$

$$= \pi r^2 \left[h - \left(\frac{2}{h}\right)\left(\frac{1}{2}h^2\right) + \left(\frac{1}{h^2}\right)\left(\frac{1}{3}h^3\right)\right] - \left[0 - \left(\frac{2}{h}\right)0 + \left(\frac{1}{h^2}\right)0\right]$$

$$= \frac{1}{3}\pi r^2 h$$

□

Proposition. The volume of a sphere formula we stated in level 2 ($\frac{4}{3}\pi r^3$) is true

Proof. Intuitively, the volume should scale with the cube of the radius. We find the volume of a semicircle rotated 2π radians about the x axis, which is a sphere. A semicircle with radius r is given by $y = \sqrt{r^2 - x^2}$ (Note: This is a semicircle and not a circle because square root is defined as just the positive square root). We then find the volume of the sphere using the volumes of revolution trick which gives the volume of a sphere with radius r as follows

$$\pi \int_{-r}^{r} (r^2 - x^2)dx$$

$$= \pi \left[\left[r^2(r) - \frac{r^3}{3}\right] - \left[r^2(-r) - \frac{(-r)^3}{3}\right]\right]$$

$$= \pi r^3 \left[\left[1 - \frac{1}{3}\right] - \left[-1 - \frac{(-1)^3}{3}\right]\right]$$

$$= \frac{4}{3}\pi r^3$$

□

Proposition. The surface area of a sphere formula we stated in level 2 ($4\pi r^2$) is true

Proof. Intuitively the surface area should depend on r^2. As r changes what is the rate at which the volume of a sphere changes, ie what is $\frac{dV}{dr}$? Visualise this scenario in your head and realize that the rate of change of volume at any instant should be the surface area, as the corresponding change in volume when we change the radius by dr is, informally, like a thingy which thickness dr, area of the surface area, and volume dV. We find that $\frac{dV}{dr}$ is $4\pi r^2$using the power rule for differentiation.

□

You can also find volumes of revolution for parametric curves, just do y^2 in place of y.

8 Differential equations

Definition. A differential equation is an equation that relates $\frac{dy}{dx}$, y and x, or more generally an equation that involves a derivative.

As an example, lets suppose we want to find all functions that equal their own derivative, which would mean all functions that satisfy the differential equation $\frac{dy}{dx} = y$. Since there is a y, we can't just integrate. We do a trick where we treat the dy and dx like a number, and in level 4 we will justify this trick in this specific context, it's not actually very difficult if you restrict to this context, but it is difficult in general to justify this kind of thing. The hand-wavy explanation is that dy is a "tiny change in y" and dx is a "tiny change in x".

So I will show this step by step to demonstrate.

Before we do this, I will note that we know that 0 equals its own derivative and so does e^x. If you think about it for a moment, you might realize that so does any multiple of e^x. It turns out that this is the most general such function. Here is how this is proven:

$$\frac{dy}{dx} = y$$

$$\frac{1}{y}\frac{dy}{dx} = 1$$

$$\frac{1}{y}dy = dx$$

$$\int \frac{1}{y}dy = \int dx$$

Note that in level 4 what we will show is that this works after integrating, as before integrating, something like $\frac{1}{y}dy = dx$ is meaningless. Apparently with some super crazy weird advanced mathematics that I don't even know as of writing this you can make it meaningful.

$$\ln(y) = x + c$$

$$y = e^{x+c}$$

$$y = e^x e^c$$

Curiously, e^c can only take values greater than 0, but something like $y = -e^x$ still satisfies the equation. Loosely speaking, this is because c does not have to be a real number for our theory to work − We will come back to this point in levels 4 and 5. We indeed get that the solution is $y = Ce^x$ where C is an arbitrary constant, and this is the most general solution.

The good news is that when you learn this, the +c starts to actually matter, so after learning this you stop forgetting it, at least that happened to me.

Often, in real life, differential equations model real world situations, where x is time and y is some variable, maybe temperature or position or whatever. In these cases, we might know what y is at the start. In the above example, if we know that $y = 1$ when $x = 0$ that is sufficient information to constrain the problem and force C to equal 1, so we have a unique solution. This is usually called a boundary condition, or an initial condition.

There is a more general formula for this: Suppose that $\frac{dy}{dx} = f(x, y)$ and it is possible to write $f(x, y)$ as a product of 2 functions that only depend on one of the variables, ie $\frac{dy}{dx} = f(x)g(y)$, then we can rearrange this (naively) and integrate both sides to get $\int \frac{1}{g(y)}dy = \int f(x)\,dx$.

Here is a concrete example:

Suppose that

$$\frac{dy}{dx} = \frac{y^2}{1+x^2}$$

$$\int \frac{1}{y^2}dy = \int \frac{1}{1+x^2}dx$$

$$-\frac{1}{y} = \arctan(x) + c$$

$$y = -\frac{1}{\arctan(x) + c}$$

Now we don't have to worry about y being 0 in the equation above since y is something which is bounded. However we do need to impose that y is not defined when $x = -\tan(c)$.

We will now solve

$$\frac{dy}{dx} = \cos^2(y)\cos(x)$$

Subject to $y = \frac{5\pi}{4}$ when $x = 0$. We get

$$\int \frac{1}{\cos^2(y)}dy = \int \cos(x)dx$$

By recalling the derivative of tan, we can antidifferentiate both sides to get

$$\tan(y) = \sin(x) + c$$

$$y = \arctan(sin(x) + c)$$

We seem to have a problem because arcan is only defined when y is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, but this is a trick. The trick is that $\tan(y) = something$ does not imply that $y = \arctan(something)$, because remember, tan is a many-to-one function. We could, however, note that if $\tan(y) = something$ then we can actually say that $y = \arctan(something) + n\pi$. This is because we can consider where the graph of $z = \tan(y)$ graph intersects with the graph of $z = something$.

So, in the problem at hand, setting n to 1 puts us in the valid range for our initial condition. We write

$$y = \arctan(sin(x) + c) + \pi$$

When $x = 0$, therefore, we want $\arctan(sin(x) + c)$ to be $\frac{\pi}{4}$. Since $\sin(0) = 0$, we can set c such that $\arctan(c) = \frac{\pi}{4}$. In other words, $c = \tan\left(\frac{\pi}{4}\right) = 1$. So $y = \arctan(sin(x) + 1) + \pi$

is our solution.

9 Numerical methods

9.1 The trapezium rule

There is also a way to approximate integrals called the trapezium rule, you essentially do this (Figure 9):

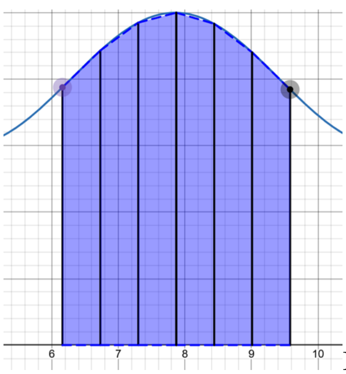

Figure 9

If your points in question are $x_0, x_1, x_2, \ldots, x_n$ and they are a distance d apart, then we add the areas of the trapeziums like

$$\frac{1}{2}d(f(x_0) + f(x_1)) + \frac{1}{2}d(f(x_1) + f(x_2)) + \cdots + \frac{1}{2}d(f(x_{n-1}) + f(x_n))$$

Which is just $\frac{1}{2}d(f(x_0) + f(x_n)) + d(f(x_1) + f(x_2) + f(x_3) + \ldots f(x_{n-1}))$ by rearranging and grouping terms.

I will not discuss this further or do any examples, I just needed to include this for completeness.

Ok, the good news is we're done with all the differentiation and integration calculus nonsense for this level.

9.2 Locating roots

Now we will talk about finding roots of functions. In general, they are difficult exactly, but there are some nice ways to get very good approximations for them.

Suppose we want to solve $x^4 - x - 1 = 0$. This has 2 real solutions. According to wolframalpha, the exact solutions look like this (Figure 10):

$$x = \frac{1}{2}\sqrt{\frac{\sqrt[3]{\frac{1}{2}(9+\sqrt{849})}}{3^{2/3}} - 4\sqrt[3]{\frac{2}{3(9+\sqrt{849})}}} - \frac{1}{2}\sqrt{4\sqrt[3]{\frac{2}{3(9+\sqrt{849})}} - \frac{\sqrt[3]{\frac{1}{2}(9+\sqrt{849})}}{3^{2/3}} + \frac{2}{\sqrt{\frac{\sqrt[3]{\frac{1}{2}(9+\sqrt{849})}}{3^{2/3}} - 4\sqrt[3]{\frac{2}{3(9+\sqrt{849})}}}}}$$

$$x = \frac{1}{2}\sqrt{\frac{\sqrt[3]{\frac{1}{2}(9+\sqrt{849})}}{3^{2/3}} - 4\sqrt[3]{\frac{2}{3(9+\sqrt{849})}}} + \frac{1}{2}\sqrt{4\sqrt[3]{\frac{2}{3(9+\sqrt{849})}} - \frac{\sqrt[3]{\frac{1}{2}(9+\sqrt{849})}}{3^{2/3}} + \frac{2}{\sqrt{\frac{\sqrt[3]{\frac{1}{2}(9+\sqrt{849})}}{3^{2/3}} - 4\sqrt[3]{\frac{2}{3(9+\sqrt{849})}}}}}$$

Figure 10

As you can see, this is horrible. We will instead find a non-exact solution.

I will plot a table of values:

x	-2	-1	0	1	2
$x^4 - x - 1$	17	1	-1	-1	13

We see, from the fact that $x^4 - x - 1$ is continuous (meaning you can draw the graph without lifting your pen), and it goes from being positive to being negative between -1 and 0, the graph MUST cross the x-axis at some point between 1 and 0.

Figure 11 shows the graph. Try to fill in the gap continuously without lifting your pen or crossin the x-axis, you can't.

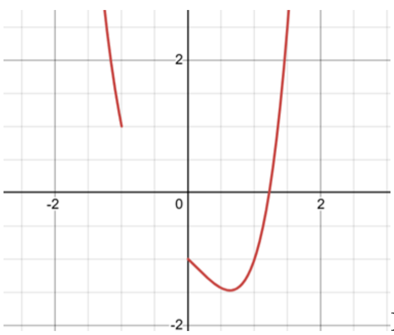

Figure 11

Interestingly, to actually say precisely what it means for a function to be continuous, and why it means it must cross the x axis like this, with a better explanation than "yeah obviously, look at it, try doing it yourself", is a surprisingly interesting question. However, we will not discuss this until later levels because the explanation I've given now should satisfy any reasonable person.

The next step would be to guess that the root is at -0.5. We can find that $(-0.5)^4 - (-0.5) - 1$ is equal to -0.4375, so we've overshot. We could keep bisecting the interval like this. We could also draw a line between the 2 points we know and find when that line crosses the x-axis and make that guess as our root, this is called linear interpolation and it is a fine strategy, however there is a better strategy. I will introduce the Newton-Raphson method.

9.3 The Newton-Raphson method

I will use this method to try to find a root of $x^3 - 3x + 1 = 0$.

First, figure 12 shows an image of the graph. We will need this:

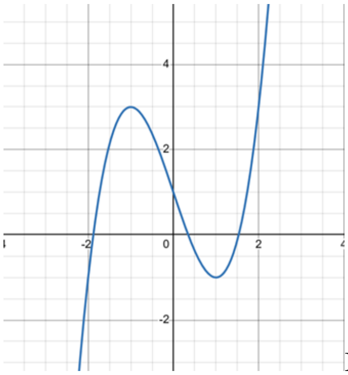

Figure 12

As you can see, 1.5 would be a good first guess for a root, however to illustrate how the method works I will use 2 as my first guess (Figure 13).

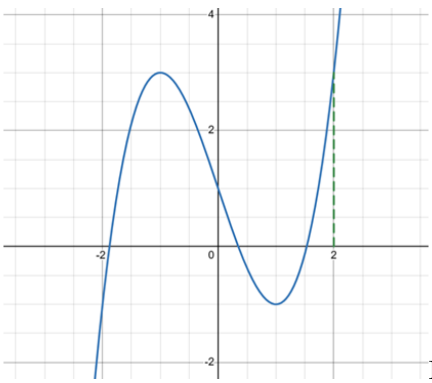

Figure 13

Now we will find the derivative of the function. It is $3x^2 - 3$. At $x = 2$, this is exactly 9. Therefore the **tangent** line has slope 15 and crosses (2,3). The equation is therefore $y = 9x - 15$. The line is shown with the graph on figure 14.

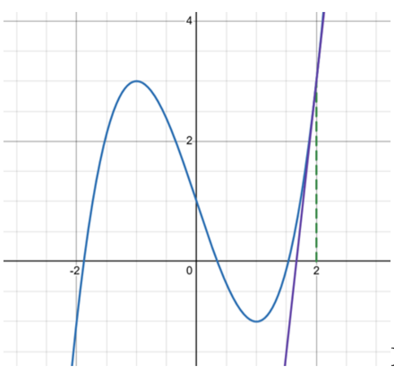

Figure 14

Now the key idea is: **The place where the tangent line crosses the x-axis is a better guess.**

We can compute that point exactly, it is $x = \frac{5}{3}$. We could find another tangent line at this point. We know that the derivative is $3\left(\frac{5}{3}\right)^2 - 3 = \frac{16}{3}$, and the function is $\left(\frac{5}{3}\right)^3 - 3\left(\frac{5}{3}\right) + 1 = \frac{17}{27}$. We could find the tangent line: It is $y = \frac{16}{3}x - \frac{223}{27}$. We could find the place this crosses the x-axis, it is $\frac{223}{144}$ which is about 1.55. Zooming in near this point looks like this (Figure 15) (The tangent line from last time is black, and the actual graph is blue):

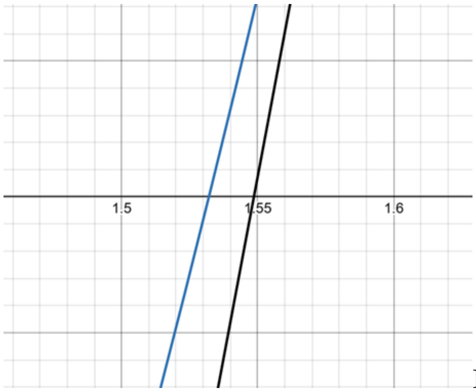

Figure 15

We see that we have zoomed in so much that the graph looks like a straight line. It therefore makes sense that if we do this again, it will be a very good guess. However, I'm too lazy and the numbers will get horrible.

Note that the newton-raphson method can fail in a few different ways. As a first example, suppose our initial guess was $x = 1$ for the above example. Then the tangent line will look like this (Figure 16). As you can see, this will never cross the x-axis.

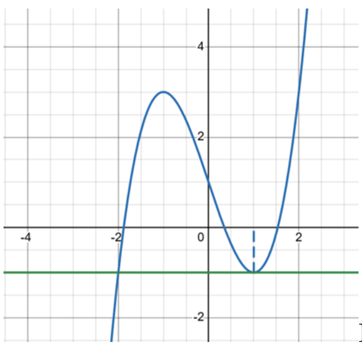

Figure 16

You could also get cases where the method is stuck oscillating between 2 points. In level 4 I show an example of this. In level 4 I also show you a few conditions where it is possible to gurantee that the method will work.

9.4 Iterative formulas

Here is another method to find roots – It's a fun one. Again, I will do it with $x^3 - 3x + 1 = 0$.

In order for this method to work, you need to rearrange the equation to $x = \textit{something}$. In the above case, it is easy, we just need $x = \frac{(x^3+1)}{3}$. First I will explain how to apply the method, then I will show you pictures to explain why it works, then show how it can fail, and in level 4 I will give 2 conditions that you can easily check for that gurantee it will work.

The method is we say $f(x) = \frac{x^3+1}{3}$, and then take an initial guess, I will pick 1, then find

1, then $f(1)$, then $f(f(1))$, then $f(f(f(1)))$, etc, to get better approximations. This is a rather weird looking method if you haven't seen it before, so I will show it to you with a picture (Figure 17). We start with x and $f(x)$ on the same axes:

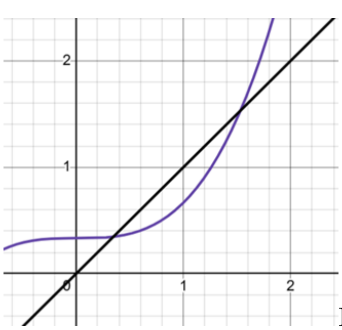

Figure 17

We note that $f(1) = \frac{2}{3}$, I will add this to the diagram (See figure 18)

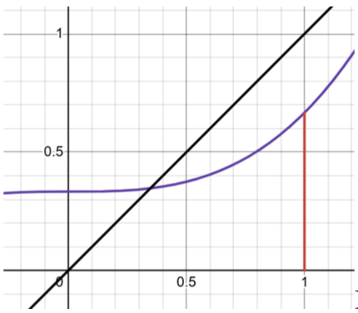

Figure 18

If we add another line, we will get to the point $(f(1),\ f(1))$ (Figure 19).

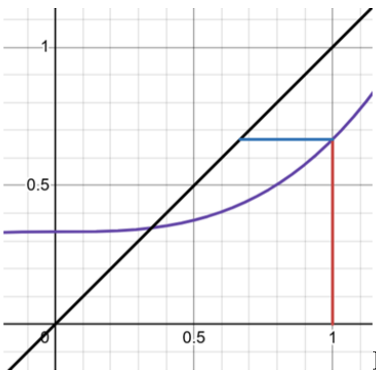

Figure 19

If we extend this line vertically downwards to the purple graph, we will reach the point $(f(1), f(f(1)))$ (Figure 20).

Figure 20

If we extend it to the black line we will reach the point $(f(f(1)), f(f(1)))$ (Figure 21).

Figure 21

Now hopefully you see the pattern. We could keep adding more lines like this and we will quickly converge to a root. This is called a cobweb diagram or a staircase diagram.

I need to actually apply this. I will calculate a few iterates numerically. You can do this with a calculator:

$$f(1) \approx 0.667, f(f(1)) \approx 0.432,\ \ f(f(f(1))) \approx 0.360,\ \ f(f(f(f(1)))) \approx 0.349$$

As promised, I will show a situation where this can fail. To do this, I will make my initial guess 2 instead of 1.

Figure 22 shows the diagram. I don't think I need to elaborate on why this will fail to converge to a root.

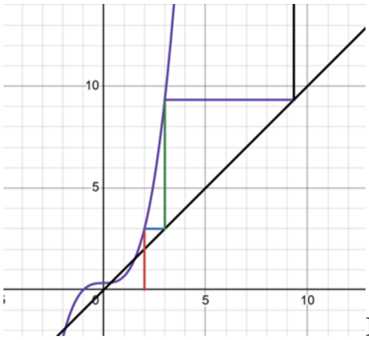

Figure 22

10 Dot products

Now, we will talk about vectors. In particular, there is an operation that takes in 2 vectors and returns a number, and it is called the dot product, or scalar product.

There are two definitions of the dot product, and it is not obvious why they are the same as eachother, and the proof of this will be in level 4. However, it is important to know both definitions, as you will see when I do examples.

The first definition: Suppose we have 2 vectors. Then the dot product between these 2 vectors is what you get if you multiply their lengths together and then multiply by the cosine of the angle between them.

As an example, lets say we want to find the dot product between the 2 vectors $a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

First, notice that if you draw a little picture it is clear that the angle between them is 45 degrees. By pythagoras, their lengths are $\sqrt{2}$ and 2 respectively. Therefore, their dot product (which we write as a.b) is $\sqrt{2} * 2 * \cos(45°) = \sqrt{2} * 2 * \frac{1}{\sqrt{2}}$, which is just 2.

The other definition of the dot product is to multiply the x coordinates, multiply the y coordinates, or (in 3 dimensions) multiply the z coordinates, or in higher dimensions multiply all of them, and then add them together. For the example above, we would get $1 * 2 \ \ (from\ the\ x) + 1 * 0 \ \ (from\ the\ y) = 2$. We see that we also get 2, so the definitions agree in this case. Like I said, in level 4 we will see that they agree in general.

Now, some crucial observations (from the cosine definition):

1. If the dot product is 0 this means that either the length of one of the two vectors is 0, or the cosine of the angle between them is 0, which means that they are perpendicular.

2. If the dot product is positive, the cosine of the angle between them is positive, so they point roughly in the same direction

3. If the dot product is ngeative, the cosine of the angle between them is negative, so they point roughly in opposing directions.

Now, consider figure 23

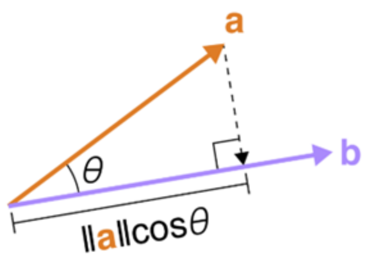

Figure 23

We see that the length shown is $|a|\cos(\theta)$ immediately from the definition of cosine. Therefore, we say that the projection of a onto b, or the amount a points in the direction of b, is this length. From the cosine definition of the dot product, this is just $\frac{|a.b|}{|b|}$. The absolute value sign on the numerator is just there to ensure that the result is positive since the length of something has to be positive.

Now consider the 2 3D vectors $a = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$. We want to find the length of the projection of a onto b. Luckily, this is easy using the formula we derived above.

First, we need to find $|b|$. By pythagoras this is $\sqrt{2^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3$.

Now we need to find $a.b$. Luckily, we have an easy formula for this, although immediately until the next level it looks like I pulled this formula out of nothing since it really does not seem related to the cosine formula. If you feel this way, know that I felt the same way when I learned this so you're in good company. We can compute $1*2+5*(-2)+7*(-1) = -15$. So using the formula, we get that the length of the projection is $\frac{|a.b|}{|b|} = \frac{15}{3} = 5$.

Next, lets take these 2 vectors and find the angle between them.

We know their dot product is -15, so $|a|\,|b|\cos(\theta) = -15$. By an application of pythagoras, $|a| = \sqrt{75}$. So we need to compute $\cos(\theta) = \frac{-15}{|a||b|} = \frac{-15}{3\sqrt{75}} = \frac{-5}{\sqrt{75}} = \frac{-5}{5\sqrt{3}} = -\frac{1}{\sqrt{3}}$. It's honestly a huge coincidence that the answer is this "nice" since I chose a randomly haha. Anyway we can get $\theta = \arccos\left(\frac{-1}{\sqrt{3}}\right)$. Using a calculator, we find that this is 125 degrees to the nearest degree.

Finally, here is one more example of a problem. Suppose we have vectors

$$\begin{pmatrix} -7 \\ 3 \\ a \end{pmatrix}, \begin{pmatrix} 3 \\ 2a \\ -5 \end{pmatrix}$$

Assume that these are perpendicular. What we want to do is find the value of a given this information.

We know that if vectors are perpendicular, then their dot product is 0. Therefore,

$$-7*3+3*2a+a*-5 = 0$$

So,

$$-21+6a-5a = 0$$

So we can find that $a = 21$, so that's our answer.