When you have a fraction with a square root on the denominator you can get rid of it ("Rationalizing" the denominator) to turn your fraction into one with all square roots on the numerator.

Example: Suppose we have $\frac{5}{\sqrt{3}}$. To solve this problem we simply multiply it by 1 in a clever way. Ie, we consider $\frac{5}{\sqrt{3}} * \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{5\sqrt{3}}{3}$.

Example: Suppose we have $\frac{8}{3-\sqrt{7}}$. This is not as straight forward. What we do is we change the minus in the denominator to a plus or if it was a plus we change it to a minus, what I mean is that our clever way of multiplying by 1 will look like $\frac{8}{3-\sqrt{7}}\frac{3+\sqrt{7}}{3+\sqrt{7}} = \frac{8(3+\sqrt{7})}{(3-\sqrt{7})(3+\sqrt{7})}$. We could expand the denominator and see that the square roots cancel, but we notice something deeper, which is that the reason we did it like this in the first place is because now the denominator looks like a thing of the form (a-b)(a+b) where in this case $a = 3, b = \sqrt{7}$, and we expand it to get $a^2 - b^2$ (difference of squares rule), which in this case is $3^2 - \sqrt{7}^2 = 9 - 7 = 2$ by definition of the square root. Thus our final answer is $\frac{8(3+\sqrt{7})}{2} = 4(3 + \sqrt{7})$. If doing a question like this, you may want to compare your answer to the original given expression using a calculator in order to verify it, and use known procedures to simplify your answer to a form a question asks for.

Recall that the quadratic formula says that if $ax^2 + bx + c = 0$ then $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. The number inside the square root $b^2 - 4ac$ says a lot. If this is a square number we can get rid of the square root and get 2 nice solutions. If it is 0 we get $x = \frac{-b\pm 0}{2a}$ so we only have 1 solution. If it is positive but not a square number we will have 2 solutions but they will stay in terms of square roots. If it is negative we will have the square root of a negative number which is not a real number, so we have no real solutions. I say real number because imaginary numbers do exist, in fact the imaginary number i is defined as the square root of -1. This is an incredibly fun area of maths that we get into in levels 4 and 5 and build on from there.

We can solve equations that are actually quadratic equations in disguise.

Example: $x^4 - 2x^2 - 3 = 0$. Here we do a change of variables which is a new concept in this website's ecosystem. We define y to be equal to x^2 and rewrite the equation as $y^2 - 2y - 3 = 0$ which has solutions -1 and 3. Therefore $x = \pm\sqrt{3}$ if we want real solutions since $y = x^2$ is -1 or 3 but the -1 case does not give real solutions.

Example: $x^7 + 5x^4 - 6x = 0$. We spot that we can factor out x and get $x(x^6 + 5x^3 - 6) = 0$, so our solutions are $x = 0$ and whatever values of x make $x^6 + 5x^3 - 6 = 0$, which we can find by doing the change of variables $y = x^3$ and solving it as in the above example.

Example: Find the values of A such that $x^2 + Ax + 3A$ has real solutions.

We can use the quadratic formula to find what those solutions would be.

We get that $x = \frac{-A\pm\sqrt{A^2-12A}}{2}$. The stuff inside the square root is called the discriminant. Note that there are real solutions for x if and only if the discriminant is non-negative.

We now need to solve $A^2 - 12A \geq 0$. Remember that to solve inequalities like this, we sketch the graph. $A^2 - 12A$ has zeroes at 0 and 12 so it will be a parabola that dips through (0,0) and goes back up through (12,0), so it will be non-negative whenever $A \leq 0$ or $A \geq 12$, and that's our answer.

Example: $\frac{(x+1)(x-2)}{x-2}$ can be simplified to just x+1 whenever x is not 2 (or else we would have $\frac{0}{0}$). However, the above is well defined for all x values that are not 2, so we can legitimately talk about the limit of $\frac{(x+1)(x-2)}{x-2}$ as x **approaches** 2. In this case, since the expression is x+1 whenever x is not 2, the limit as x approaches 2 in 3. The graph basically looks like the graph of x+1 with a hole at x=2. The propert way to write this is $\lim_{x \to 2} \frac{(x+1)(x-2)}{x-2} = 3$.

The precise definition of this is:

$\frac{(x+1)(x-2)}{x-2}$ gets and stays as close to 3 as we want if x is sufficiently close to 2. It does not need to just get closer to the value we want (otherwise the sequence 2.1, 2.01, 2.001 would approach 1 but we want to say it approaches 2), it has to get as close as we want no matter how close we set this threshold.

If we define $f(x)$ as x-1 if x is negative and x+1 if x is positive, the graph will look as below

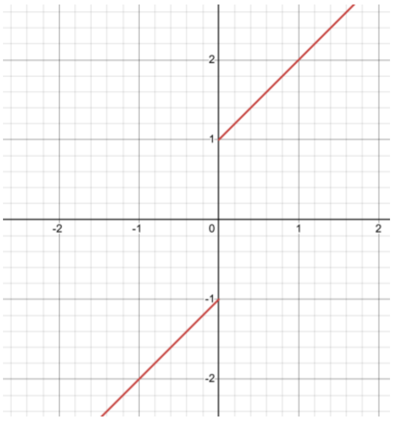


Image: Graph of f(x)

The limit of f(x) as x goes to 0 is not defined since we can make x get as close to 0 as we want but f(x) will not get as close as we want to some value – It will either be near -1 or near 1 depending on if x is positive or negative. However, we can write this as follows:

$$\lim_{x \to 0^+} f(x) = 1$$

means the limit as x goes to 0 from the right. Similarly we have

$$\lim_{x \to 0^-} f(x) = -1$$

We can also have limits as x goes to infinity of something. As an example, if $g(x) = \frac{1}{x}$ then since as x goes to infinity g(x) approaches 0, we write

$$\lim_{x \to \infty} g(x) = 0$$

We can also have limits of things as x goes to negative inifinity, in this case it will also be 0.

In the case we have an infinity, we replace "x is sufficiently close to infinity" in the definition of a limit with "x is large enough".

So that's a brief introduction to limits, now we're ready to talk about what a derivative is.

We have seen how to find the slope of a line. Now imagine we have a curve and we want to find the slope of the curve.

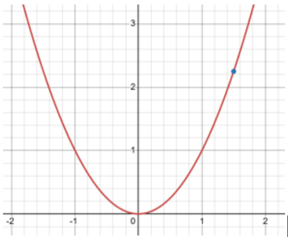
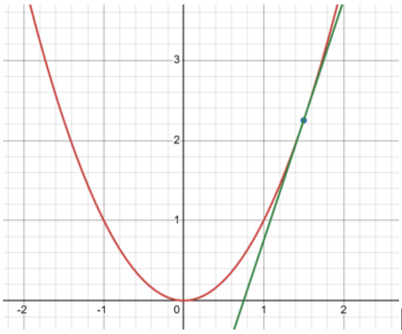
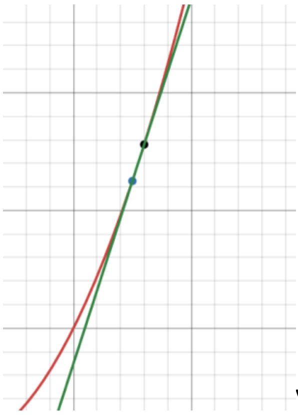


Image of a parabola $y = x^2$: Suppose we want to find its slope at this blue point.


I have added a green line to the image above that is tangent to the parabola at the blue point which is (1.5, 2.25). It looks like the slope is about 3 (the green line moves up 3 squares for each square it moves to the right). The question is: Is the slope exactly 3? How do we find this tangent line? How do we prove the slope is 3?

The idea is we want to approximate the slope. To do this, imagine taking a point very close to the blue point on the parabola, such as (1.6, 2.56) (2.56 because $y = x^2$ so the y coordinate is 1.6^2 which is 2.56 and the point must be on the parabola) and making a line that goes through those 2 points. We know how to find an equation of the line through $(1.5, 2.25)$ and $(1.6, 2.56)$ from the previous level, and it turns out that this is $y = 3.1x - 2.4$, so our approximation for the slope is 3.1.


Visualisation of this, as you can see it is a pretty good approximation.

Note that we cannot do a perfect approximation – if our 2 points were $(1.5, 2.25)$ and $(1.5, 2.25)$ there are infinitely many lines through that singular point so it is not much use. But we can do a better approximation: Lets find the slope of the line through $(1.5, 2.25)$ and $(1.501, 2.253001)$. I won't go through the calculation but long story short the slope is 3.001, so it really does seem like it is approaching 3.

Now if you see where this is going, we are indeed going to take a limit. Imagine increasing 1.5 by a very small amount like dx, then we want to find the limit of the change in y divided by the change in x as dx goes to 0. Ie, we want $\lim_{dx \to 0} \frac{(1.5+dx)^2 - 1.5^2}{dx}$. So lets try to calculate this limit.

$$\lim_{dx\to 0}\frac{(1.5+dx)^2-1.5^2}{dx}=\lim_{dx\to 0}\frac{1.5^2+3dx+dx^2-1.5^2}{dx}=\lim_{dx\to 0}\frac{3dx+dx^2}{dx}=\lim_{dx\to 0}3+dx=3$$

So the slope is indeed 3 like we guessed.

In general, if we have a function f(x), we define the derivative of f at c as the slope, which is defined by:

$$\lim_{dx\to 0}\frac{f(c+dx)-f(c)}{dx}$$

This limit will exists if the function is smooth at that point, if it had a spike there it would not have a well defined slope and this limit would not exist. A function where this limit exists is called differentiable.

We write the derivative of f(x) as $f'(x)$ or $\frac{d}{dx}f(x)$. Also, if the function is y as a function of x that we would graph, we write the derivative as $\frac{dy}{dx}$.

If we replace “1.5” with “x” and do the same derivation as above we will see that it is always the case that the parabola $y = x^2$ has a slope of 2x at each value of x. This is because we did the algebra and got

$$(x+dx)^2-x^2=x^2+2xdx+\cdots-x^2=2xdx+\cdots$$

noting that powers of dx above 1 will vanish (exactly like in the example above). We will show more expansions of things and spot a pattern:

$$(x+dx)^2-x^2=2xdx+\cdots$$

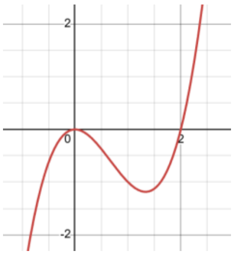
$$(x+dx)^3-x^3=3x^2dx+\cdots$$

$$(x+dx)^4-x^4=4x^3dx+\cdots$$

In general this pattern holds: The slope of x^n is nx^{n-1}. The proof of this for the case n is an integer is immediate from the binomial theorem (a later topic which we prove for the positive integer case in this level). We can prove this for all n including non-integers either using a clever method that relies on later topics or using the generalized binomial theorem but that version is much harder to prove, and therefore all this discussion will be deferred to the next level (level 4) where I actually show how all og these proofs would work and prove that a non-integer power is actually well defined. For now we will assume the above rule to be true for all n. This only works if n is a number, it does NOT work to find the slope of x^x, an example we will come back to when we develop more theory.

Example: We can find the slope of $x^3 - 2x^2$ using the fact that slopes add and using the rule above. This gives $3x^2 - 4x$ as our answer. Note that slopes do not multiply in the same way they add – we will see later how to deal with this, but do not make any assumptions and make sure to apply all the rules we will derive carefully.

Now lets put an image of what the graph of $x^3 - 2x^2$ actually looks like.



Now we have all the theory we need to work out exactly where the point is that the graph becomes flat: It is when the derivative is 0. Therefore we need to solve the equation $3x^2 - 4x = 0$. The solutions to this equation are 0 and $\frac{4}{3}$, so it is exactly when x is 0 and x is $\frac{4}{3}$ that the graph is flat and turns around, consistent with the image above.

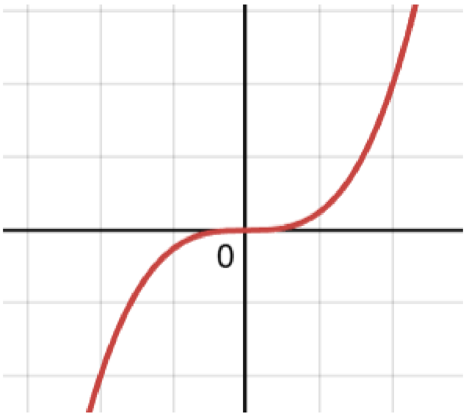
Example of why this is useful: We can solve a problem such as: A rectangle has a perimeter of 20 units. What is its maximum area?

Lets write down what we know: Its perimeter (the total length of its edges) is 20 meters, which is twice the short side plus twice the long side, so the short side plus the long side is 10. Therefore if one of the side lengths is x, the other side length is 10-x since they add up to 10. We now know that the area of the rectangle is given by x(10-x) (One side length times the other side length) which expands to the expression $10x - x^2$. This is differentiable everywhere with derivative 10-2x. Therefore if it is at its maximum, either its slope must equal 0 or it must be at one of its extreme values (when x=0 or x=10), but in these cases the area is 0, so we will solve for when the slope is 0. The equation 10-2x=0 is easy to solve, the answer is x=5. So the area at this point is $10 * 5 - 5^2 = 25$, so the maximum area is 25 and that's our answer.

Note that the point where the slope of 0 corresponds to a maximum instead of a minimum for 3 reasons:

1. x=5 is the only possible turning point of $10x - x^2$ and at values of x around x=5 (such as 0 and 10) the value of $10x - x^2$ is smaller than 25, so if it was, say larger than 25 at x=6, it would have to turn around again to get to the point (10, 0) which we know it does not.
2. The slope (10-2x) is negative to the right of x=5 and positive to the left of x=5.
3. $10x - x^2$ is a downward facing parabola so its only "flat" point is a local maximum.

A point where the derivative is 0 is called a stationary point. We will talk more about how to determine the nature of these like we did in the example above. They can be local minimums, local maximums, or neither (such as $y = x^3$ at (0,0) seen in the image below).



Fun fact: A parabola is the trajectory of a ball if you throw it and assume there is no air resistance.

Intuition for what we did above: We cannot really determine how fast something is going by taking a picture of it, but we can take a picture of it at two nearby moments in time and take the distance it moved divided by the time between the pictures to get a decent approximation. This is how things that measure speed work in real life.

Note that we can use the derivative to analyze a function in the sense that we know it is increasing when the derivative is positive and decreasing when the derivative is negative.

Now we can find tangents to graphs like I did in the example above. Lets do the example with the graph $y = x^2$ at $(1.5, 2.25)$. We know the tangent will be a line of slope 3 passing through the point $(1.5, 2.25)$. To find such a line, note that a valid equation for the line (something you can do more generally for problems where you know the slope and a point the line passes through is)

$$y - 2.25 = 3(x - 1.5)$$

Doing some algebra and rearranging the equation above, $4y - 12x + 9 = 0$ is the equation for our tangent line.

We can also find the normal to $y = x^2$ at $(1.5, 2.25)$, which means the line perpendicular to the curve at that point. To do this recall from last levelthat the line perpendicular to a line with slope m has slope $-\frac{1}{m}$, so our normal will pass through $(1.5, 2.25)$ and have slope $-\frac{1}{3}$. So the equation is

$$y - 2.25 = -\frac{1}{3}(x - 1.5)$$

Again we can rearrange this:

$$y + \frac{1}{3}x - \frac{11}{4} = 0$$

Multiply by all denominators and get rid of common factors to turn this into integers:

$$12y + 4x - 33 = 0$$

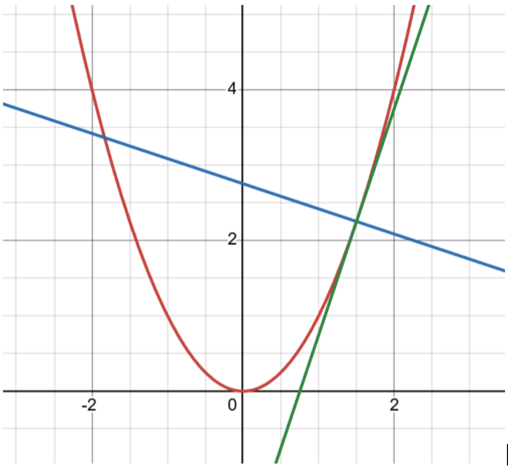


Image of the graph with the tangent and normal added in.

We will now talk more about powers and try to understand what their graphs will look like. For example what the graph of $y = 2^x$ might look like.

We note that $y = 2^x$ increases as x increases, and we will give a table of some values:

X=-3	X=-2	X=-1	X=0	X=1	X=2	X=3
1/8	1/4	1/2	1	2	4	8

We will now show what the graph looks like.

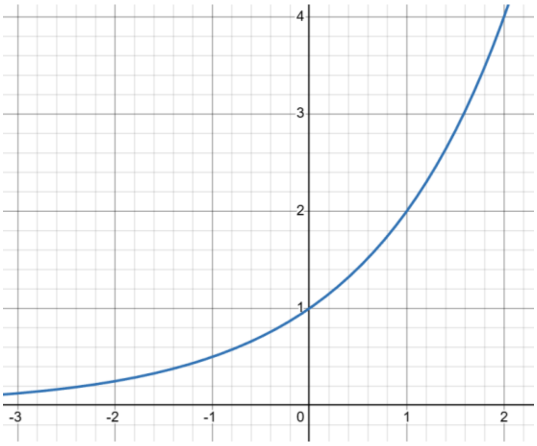


Image: Graph of $y = 2^x$

We see that the graph grows quite fast. Intuitively, powers have the property that they grow as fast as their value, leading to runaway growth, or exponential growth. This is exactly what happens with the spread of diseases or with interest in a bank. This leads to the natural question of when we have something whose derivative is equal to the function itself causing the function to increase in a runaway effect like this.

It turns out that there is a special number called e which is about 2.71828182845904523536... with the property that $y = e^x$ has a derivative equal to e^x.

Now here is another natural question:

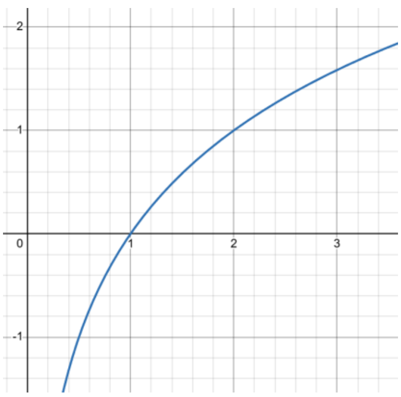
If we have to solve for x in $a^b = x$ we can simply work out a^b in a calculator. In order to solve an equation like $x^a = b$ we just need the a'th root of b. Now the question is what if the exponent is unknown, ie we want to solve $a^x = b$. Then this is where we introduce logarithms. The definition of a logarithm is as follows:

If $a \neq 1, a > 0, b > 0$ to ensure the graph of a^x is one-to-one and hits b, then we define $\log_a(b)$ to be the unique real solution x to $a^x = b$.

Example: $\log_{10}(10000) = 4$ because $10^4 = 10000$, so this essentially gives you a rough estimate of how many digits are in the numbers.

Definition: $\ln(x) = \log_e(x)$. This is called the natural logarithm. Often people just write $\log(x)$ which means $\log_{10}(x)$ if you are a physicist or engineer but for us it means $\ln(x)$ because we are mathematicians.

$\log_a(x)$ as a function from the positive numbers to the real numbers is an inverse to the function a^x from the real numbers to the positive numbers.



As you should know by properties of graphs of inverse functions, this image is what the graph of $y = \log_2(x)$ looks like.

We need some basic rules of logarithms.

$$\log_a(b) + \log_a(c) = \log_a(b * c)$$

Before I prove this, note that the base is constant, and that it is NOT the case that

$$\log_a(b) + \log_a(c) = \log_a(b + c)$$

The additive property above is 99% of the time not true for functions so if this idea is in your head you need to eliminate it.

So the proof is as follows:

$a^{\log_a(b)} = b$ by definition. Therefore $a^{\log_a(b)+\log_a(c)} = a^{\log_a(b)}a^{\log_a(c)} = bc = a^{\log_a(bc)}$. Since a^x is a one to one function we can cancel it from both sides to get the rule.

Similarly,

$$\log_a(b) - \log_a(c) = \log_a\left(\frac{b}{c}\right)$$

Similarly

$$k\log_a(b) = \log_a(b^k)$$

This is because $a^{k\log_a(b)} = \left(a^{\log_a(b)}\right)^k = b^k = a^{\log_a(b^k)}$

Also

$$\log_a(1) = 0$$

Because $a^{\log_a(1)} = 1 = a^0$

By definition $\log_a(a^k) = k$ and therefore $\log_a(a) = 1$.

The final rule we will discuss is

$$\frac{\log_c(x)}{\log_c(a)} = \log_a(x)$$

The proof is as follows.

$$c^{\log_c(x)} = x = a^{\log_a(x)} = \left(c^{\log_c(a)}\right)^{\log_a(x)} = c^{\log_c(a)\log_a(x)}$$

Cancelling c from the left and right hand sides and rearranging gives the desired result. These rules can be used to rearrange expressions involving logs. Numerical values of logs should be calculated using a calculatro unless you can spot the answer like in $\log_{10}(10000) = 4$.

Example: Lets solve $2^x = 5$. The solution is $x = \log_2(5)$.

Example: Lets solve $9^x - 12 * 3^x + 27 = 0$. We can put everything on the same base and use some power rules to realize this is one of those secretly quadratic equations.

$$(3^x)^2 - 12(3^x) + 27 = 0$$

So the values of 3^x that satisfy this are 3 and 9, which conveniently give x values of 1 or 2. One time on my A level exam I stupidly forgot to actually unwind at the end to get x and lost some marks for that on a question very similar to this.

Now we define two rules for differentiation:

The product rule says that $\frac{d(f(x)g(x))}{dx} = g(x)\frac{df(x)}{dx} + f(x)\frac{dg(x)}{dx}$.

Example:

$$\frac{d(x^2e^x)}{dx} = e^x\frac{d}{dx}x^2 + x^2\frac{d}{dx}e^x = e^x 2x + x^2e^x = e^x(x^2+2x)$$

Proof of the product rule:

$$\frac{d}{dx}f(x)g(x)$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

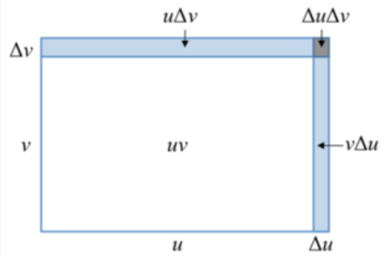
$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - f(x)] \cdot g(x+\Delta x) + f(x) \cdot [g(x+\Delta x) - g(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot \lim_{\Delta x \to 0} g(x+\Delta x) + \lim_{\Delta x \to 0} f(x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(x)g(x) + f(x)g'(x).$$

The fact that $\lim_{\Delta x \to 0} g(x+\Delta x) = g(x)$ follows from the fact that differentiable functions are continuous.



Geometric illustration of a proof of the product rule[1]

This proof contains an image with a visual proof of the product rule.

In the geometric illustration above u is shorthand for f(x) and v is shorthand for g(x).

The chain rule says that $\frac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right)g'(x)$.

We can get the quotient rule for $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x) * \frac{1}{g(x)}\right) = \frac{d}{dx}\left(f(x) * h\left(g(x)\right)\right)$ where $h(x) = \frac{1}{x}$ and then carefully apply the product and chain rules. We get $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$.

Example:

$\frac{d}{dx}\left(e^{(x^2)}\right)$ where here g is x^2 and f is e^x, applying the rule carefully gives

$$\frac{d}{dx}\left(e^{(x^2)}\right) = 2x\left(e^{(x^2)}\right)$$

Intuition for the chain rule:

It basically says $\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$ where dx means “tiny change in x”, dg means “resulting tiny change in g”, and df means “resulting tiny change in f”. A more formal proof is given in level 4.

Example: Suppose we want to differentiate 2^x. We do this using the chain rule.

$2^x = \left(e^{\ln(2)}\right)^x = e^{x\ln(2)}$ so

$$\frac{d}{dx}(2^x) = \frac{d}{dx}\left(e^{(x\ln(2))}\right) = \ln(2)\, e^{x\ln(2)} = 2^x\ln(2)$$

By the chain rule. In general,

$$\frac{d}{dx}(a^x) = a^x \ln(a)$$

For all positive a where its natural logarithm is defined.

Definition: The second derivative of a function, written as $\frac{d^2 f(x)}{dx^2}$ or $f''(x)$ or $f^{(2)}(x)$ where the latter notation is annoying as it can mean do f twice, take f squared, or take f's second derivative, is the derivative of the derivative. Note that if it is positive, the derivative is increasing, so the function is curving upwards. If it is negative the derivative is decreasing so the function is curving downwards. A point of inflection is a point where the second derivative is 0. Using the sign of the second derivative can allow you to easily determine whether a stationary point is a local minimum or local maximum by considering the curvature type.

We can similarly define third derivatives and higher order derivatives.

Before we move onto integrals I will briefly introduce the absolute value. For real numbers, it is simple: For positive numbers don't do anything, for negative numbers just remove the minus sign. The notation for this is to put vertical bars on each side of the number.

Example: |5|=5, |0|=0, |-3|=3

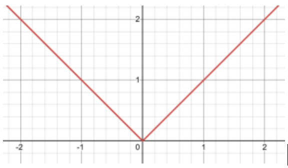


Image: Graph of y=|x|

We define an integral as a reverse derivative. We write $\int f(x)dx$ to mean a function such that when we differentiate it or find its slope we get f(x). However, note that there is not a unique such function. For example, 7x, 7x+5, 7x-3, 7x+100000, and 7x+π all have a slope of 7, but these only change by adding a constant. Therefore, if we compute an integral, we always put "+c" at the end to note that we can add an arbitrary contant.

We will derive some rules to help you integrate stuff.

First,

$$\int nx^{n-1}dx = x^n + c$$

By the reverse power rule for differentiation. It therefore follows that,

$$\int (n+1)x^n dx = x^{n+1} + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

Ie, to integrate x to the power of a number, we increase the power by 1 and divide by the new power. This is only true if x is a number, not if it is a non-constant function.

IMPORTANT: We are ONLY allowed to divide by n+1 as above because it is a FIXED number. In general, we cannot just multiply and divide things outside of integrals like that.

Example:

$$\int x^2 dx = \frac{x^3}{3} + c$$

$$\int x^3(x+\sqrt{x})dx = \int x^3\left(x^1 + x^{\frac{1}{2}}\right)dx = \int x^4 + x^{\frac{7}{2}}dx = \frac{x^5}{5} + \frac{x^{\frac{9}{2}}}{\frac{9}{2}} + c = \frac{1}{5}x^5 + \frac{2}{9}x^{\frac{9}{2}} + c$$

Since things in integrals add nicely – to see this, note that adding things adds their slope and adding their slope adds what thing that was the slope of.

Final example:

$$\int \frac{1}{x}dx = \int x^{-1}dx = \frac{x^0}{0} + c$$

Oh no! We divided by 0. The case $\int \frac{1}{x}dx$ is special and we will come back to it later. it has a nice but possibly surprising answer.

We also note that, by direct reverse differentiation,

$$\int a^x(\ln(a))dx = a^x + c$$

Therefore,

$$\int a^x dx = \frac{a^x}{\ln(a)} + c$$

This is true for positive a unless a=1 where we divide by 0, but that has a straight forward answer:

$$\int 1^x dx = \int 1dx = x + c$$

This is another example of a situation where in a division by 0 case we have an exception to the rule. With more advanced techniques which we develop in upcoming levels, you can show that these exception cases are the limit of “nearby” cases so the behavior does not jump suddenly.

We now consider definite integrals. Here is how we do this. I will explain how to compute it first and then explain its significance after that.

Definition: We say a function is continuous if you can draw its graph without taking your pen of the paper, or equivalently/semi-formally the output stays arbitrarily close to the output at a point when you are sufficiently close to that point

If I have a number on the bottom and top of the integral sign, like this,

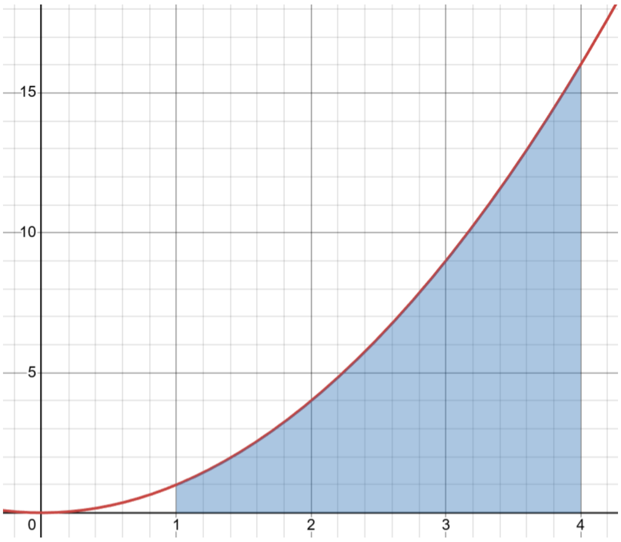
$$\int_1^4 x^2\, dx$$

Then what I do is as follows (provided the function does not blow up to infinity between 1 and 4 inclusive and is either continuous or made of continuous parts – all A level questions satisfy this except “improper inetegrals” which we cover in level 5):

1. Find the “antiderivative”. In this case it is $\frac{x^3}{3}$
2. Evaluate the antiderivative when x=1 and x=4 because these are the numbers at the bottom and top. In this case we have $\frac{1^3}{3} = \frac{1}{3}$ and $\frac{4^3}{3} = \frac{64}{3}$
3. Subtract the result using the top number from the result using the bottom number. In this case we get that $\frac{64}{3} - \frac{1}{3} = \frac{63}{3} = 21$. Therefore we have our answer:

$$\int_1^4 x^2 \, dx = 21$$

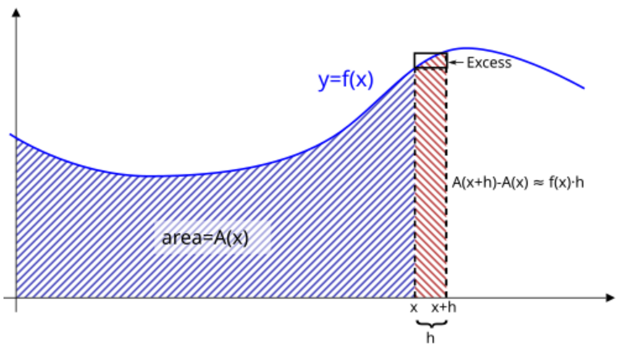
But this leaves many questions, the first of which is why do this? The short answer is it gives you the **area under the curve** of x^2 from 1 to 4, ie the area of the blue region in this image:



We will address shortly why it gives the case and what happens if the graph goes under the x-axis. First, we will address why the method above is well defined.

We say f is the function we are integrating and a and b are the bounds of the integral (like 1 and 4 in the above example) and F is its antiderivative, so we would find F(b)-F(A). However, first we need to show that F(b)-F(a) is well defined, since F(x) is not *the* antiderivative of f, rather *an* antiderivative. The family of antiderivatives of f(x) is given by F(x)+c. But, notice that in (F(b)+c)-(F(a)+c) the c’s cancel, so as long as we evaluate the difference between an antiderivative when evaluated at b and a, we will get the same value regardless of which antiderivative we use, as long as we are consistent with which antiderivative we use whether we evaluate at a or b.

The fact that this procedure gives the area is called the **Fundamental theorem of calculus**. Calculus is the branch of maths that studies these derivatives and integrals and stuff. We give an informal derivation here of why this is true and prove it from more precise definitions of the integral in level 4.



There is another way to *estimate* the area of this same strip. As shown in the accompanying figure, h is multiplied by $f(x)$ to find the area of a rectangle that is approximately the same size as this strip. So:

$$A(x+h)-A(x)\approx f(x)\cdot h$$

Dividing by h on both sides, we get:

$$\frac{A(x+h)-A(x)}{h}\approx f(x)$$

This estimate becomes a perfect equality when h approaches 0:

$$f(x)=\lim_{h\to 0}\frac{A(x+h)-A(x)}{h}\overset{\text{def}}{=}A'(x).$$

The
funct
two

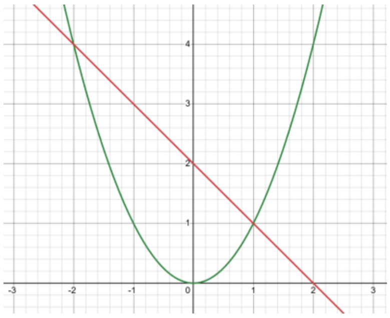
Image to show why FTC holds intuitively.

Note: A(x) is the area from a starting point, it doesn't matter which starting point we pick, but it should not be negative infinity like my A level textbook does, since otherwise A(x) would not always be well defined!

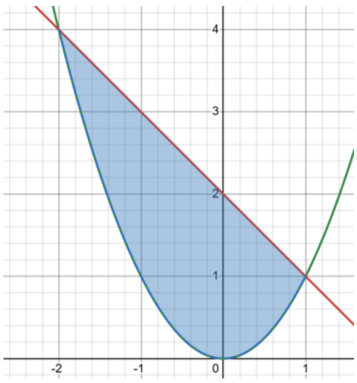
Now one alternative way of stating the fundamental theorem of calculus is to say that f(x) is the derivative of $\int_a^x f(t)dt$ which is equal to F(x)-F(a) where F(x) is an antiderivative of f(x). Since F(a) is a constant, we have that the derivative of F(x)-F(a) equals f(x).

Note that by the proof above, we see that if the graph goes under the x-axis, we subtract the area enclosed by the graph and the x-axis at those points.

It is possible to find the area enclosed by two graphs. We will do the example in the image below of the graphs $y=x^2$ and $y=x-2$



Specifically, we want the area of the blue region in the following image:



If you're an engineer, just count the squares. Otherwise, follow the steps below:

Step 1: Find the intersection points.

We want $x^2 = x - 2$. This is a familiar quadratic equation and its roots are x=1, x=-2, agreeing with the diagram.

Step 2: Convert to integrals

Now we note that the blue area is exactly the difference between the following areas gometrically, represented as an integral:

$$\int_{-2}^{1}(2-x)dx - \int_{-2}^{1}x^2dx$$

So lets compute this.

Step 3: Find the antiderivatives

They are $2x - \frac{x^2}{2}$ and $\frac{x^3}{3}$ by standard antiderivative facts derived above. As discussed, the +c does not matter as it will cancel shortly.

Step 4: Compute and simplify carefully with all minus signs in check

We have

$$\left[\left(2*1-\frac{1^2}{2}\right)-\left(2*-2-\frac{(-2)^2}{2}\right)\right]-\left[\left(\frac{1^3}{3}\right)-\left(\frac{(-2)^3}{3}\right)\right]=\left[\frac{3}{2}-(-6)\right]-\left[\frac{1}{3}-\left(-\frac{8}{3}\right)\right]=\left[\frac{15}{2}\right]-[3]$$
$$=\frac{9}{2}=4.5$$

We could do more complicated problems where the bounds are the axes, or tangents or normals to curves, by similar methods.

We will now try to find the derivative of $\ln(x)$. The chain rule implies that we can say $\frac{dx}{dy}\frac{dy}{dx} = 1$ when both derivatives exist (and therefore both non-zero so that the other one exists). So suppose that we know $y = \ln(x)$ (x>0), then we can say $x = e^y$ and therefore $\frac{dx}{dy} = e^y$ which always exists and is never 0. We now know that $\frac{dy}{dx} = \frac{1}{e^y}$ by the chain rule so $\frac{dy}{dx} = \frac{1}{x}$. Conveniently, this resolves the problem of integrating $\frac{1}{x}$ from earlier. However, this is only for x>0. For negative x values it is somewhat contraversial.

Basically, by symmetry of the $\frac{1}{x}$ graph, we know that $\ln(-x)$ is a perfectly valid antiderivative of $\frac{1}{x}$ for x<0, and therefore the antiderivative is often said to be $\ln(|x|) + c$. However, it is perfectly valid to think of it as $\ln(x) + c$ where in the x<0 case it is just $\ln(x) + \ln(-1) + c$ where $\ln(-1)$ is a sort of "virtual number", and this is still $\ln(x) + c$ where c is any constant. In later levels we will understand these "virtual numbers" better and even see a connection to imaginary numbers which are related to the square root of -1. In fact, foreshadowing, we will see that $\ln(-1) = \pi * \sqrt{-1}$ which seems magical. Either way is consistent as long as your integration bounds are on the same side as the problematic x=0 point, as I mentioned that you need to have the integral not blow up in the range – If we allow this, we could argue that something like $\int_{-1}^{1}\frac{1}{x}dx$ is 0 by symmetry of the graph or $\ln(-1)$ by the trick above, and this is a contradiction. This comes from the fact that none of the antiderivatives of $\frac{1}{x}$ are continuous at x=0 which can happen since $\frac{1}{x}$ blows up near there.

Tricky example: Lets differentiate x^x. First of all, DO NOT USE THE POWER RULE. I have made it clear that the power rule is only for x to the power of a CONSTANT. We need a trick. If you see any exponent that is not of any of the forms you learned to differentiate above, do something like this logarithm/chain rule trick presented below.

$$\frac{d}{dx}x^x = \frac{d}{dx}\left(e^{\ln(x)}\right)^x = \frac{d}{dx}e^{xln(x)} = e^{xln(x)}\frac{d}{dx}\left(xln(x)\right)$$

Where in the last step we used the chain rule. We can now use the product rule:

$$\frac{d}{dx}x^x = e^{xln(x)}\frac{d}{dx}\left(xln(x)\right) = e^{xln(x)}\left[\ln(x)\frac{d}{dx}x + x\frac{d}{dx}\ln(x)\right] = e^{xln(x)}(\ln(x)+1)$$

Recalling from earlier that $e^{xln(x)} = x^x$, our final correct answer is

$$\frac{d}{dx}x^x = x^x(\ln(x)+1)$$

I will now do an example ofa problem involving connected rates of change.

A circle grows such that its radius increases at a rate of 6 meters per second. Find the rate of change of its area when its radius is 7 meters.

Solution: $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$. We want $\frac{dA}{dt}$ which is (chain rule) $\frac{dA}{dr}\frac{dr}{dt} = 2\pi r * 6$. When r=7 this is 84π, so that's our answer.

In general you need to apply this kind of principle to solve problems.

Going back to quadratics, we can go beyond quadratics.

Definition: A polynomial is an expression such as $5x^3 + 7x^2 - x + 9$, ie x to the power of positive integers multiplied by stuff and added together. A root of a polynomial is a value of x such that this equals 0. The degree of a polynomial is the highest power that appests – 3 in this case.

Example: Consider $x^3 - 3x^2 + 4x - 4$. We can divide this by $x - 2$ using a long division algorithm, here is how this would work:

Since we have an x^3 term, to get this term we would need x^2 lots of $x - 2$, and this would give us $x^3 - 2x^2$ so we would have $-x^2 + 4x - 4$ left over. To get the leftmost term we try $-x$ lots of $x - 2$ which is $-x^2 + 2x$ then what we have left is $2x - 4$ which is 2 lots of $x - 2$. Therefore our factorization is as follows:

$$x^3 - 3x^2 + 4x - 4 = (x-2)(x^2 - x + 2)$$

It follows that since x-2 factors into this exactly, 2 is a root of $x^3 - 3x^2 + 4x - 4$ as when you plug in 2 to the expression you get 0. It turns out that x-A factoring into the polynomial without remainder like above (more on this below) and A being a root are equivalent statements and this is called the factor theorem, something we will discuss more and rigorously justify in level 4.

Common factors may be cancelled out in fractions where you have one polynomial divided by another. However, it may not always factor so nicely, we may get a remainder term, for example, adding 1 to the last example, trying to divide $x^3 - 3x^2 + 4x - 3$ by $x - 2$ will give a remainder term of $\frac{1}{x-2}$ and we say the remainder is 1.

The remainder theorem states that if a polynomial is f(x) then f(A) is the remainder when you try to divide it by x-A. For example, if $f(x) = x^3 - 3x^2 + 4x - 3$ then when x=2, f(x)=1 as the remainder of f(x) was 1 when we divided it by x-2. We rigorously justify this in Level 4, the arguments for the factor and remainder theorems are not difficult but we postpone them to Level 4.

Using the factor theorem we can solve equations by spotting a root.

Example: Given that x=2 is a solution to $x^3 - x^2 - 32x + 60 = 0$ find all the solutions

By the factor theorem we factor out x-2 with long division. That gives $(x-2)(x^2 + x - 30) = 0$ as an equivalent equation to solve. We can factor quadratics to get $(x-2)(x-5)(x+6) = 0$ and therefore x is 2, 5 or -6.

Example: Solve $x^3 - 3x + 2 = 0$. It is easy to see that 1 is a root (1-3+2=0) so we can proceed as above (factor out x-1) and we get $(x-1)(x^2 + x - 2) = 0$ so $(x-1)(x-1)(x+2) = 0$ so x=1 or x=-2 is our solution.

If you divide a polynomial by a polynomial of degree d you can have a remainder with up to degree d-1. There is a thing you can do with remainder-like terms by factoring the denominator. Suppose you can factor it into linear factors.

Example:

We must simplify $\frac{x^2+7}{(x+1)(x-2)(x+3)}$. The numerator has lower degree than the denominator so we cannot long divide to get any non-remainder terms. We can write it as $\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+3}$ where A, B and C are to be found. Although we can solve it using the method I will show and it will work in each case, the proof that it works in every case (including the special case outlined below) is deferred to level 4. The way to do it is to get a common denominator.

$$\frac{A(x-2)(x+3)}{(x+1)(x-2)(x+3)} + \frac{B(x+1)(x+3)}{(x+1)(x-2)(x+3)} + \frac{C(x+1)(x-2)}{(x+1)(x-2)(x+3)}$$

We then have

$$A(x-2)(x+3) + B(x+1)(x+3) + C(x+1)(x-2) = x^2 + 7$$

Expanding and regrouping terms,

$$(A + B + C)x^2 + (A + 4B - C)x - 6A + 3B - 2C = x^2 + 7$$

We now have to solve the equations

A+B+C=1

A+4B-C=0

-6A+3B-2C=7

Carrying out this annoying process we will get that $A = -\frac{4}{3}, B = \frac{11}{15}, C = \frac{8}{5}$

If the denominator has a repeated root (meaning a factor like $(x-a)^n$ with n more than 1 appears), we get one term with all the powers, as in the example below:

$$\frac{x^4+2x^3+1}{x^2(x-1)^3(x-7)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3} + \frac{F}{x-7}$$

I'm not gonna solve this equation, in practice you will probably just need to do something like a linear term squared times a different linear term.

This is useful for integration. Example (using the previous example):

$$\int \frac{x^2+7}{(x+1)(x-2)(x+3)}dx = -\frac{4}{3}\int \frac{1}{x+1}dx + \frac{11}{15}\int \frac{1}{x-2}dx + \frac{8}{5}\int \frac{1}{x+3}dx$$
$$= -\frac{4}{3}\ln(x+1) + \frac{11}{5}\ln(x-2) + \frac{8}{5}\ln(x+3) + c$$

A bit more stuff on vectors:

A unit vector is a vector with length 1. Vectors are parallel if they point in the same direction. They are perpendicular if the angle between them is 90 degrees. A line can be thought of a point on the line plus a multiple of a vector.

Example:

The lines

$\begin{pmatrix}1\\2\\3\end{pmatrix} + a\begin{pmatrix}4\\5\\6\end{pmatrix}$ and $\begin{pmatrix}7\\8\\9\end{pmatrix} + b\begin{pmatrix}10\\11\\13\end{pmatrix}$

Suppose that the numbers in the vector are the x-coordinate, y-coordinate and z-coordinate (height) respectively. We will often call these components.

Never intersect, because

For the x and y coordinates to match $1+4a=7+10b$ and $2+5a=8+11b$ implies a=-1, b=-1, but that is incompatible with the z coordinates also matching up.

Example:

Suppose

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}A\\6\\7\end{pmatrix}, \begin{pmatrix}7\\10\\B\end{pmatrix}$$

Lie on the same line and we want to find A and B, then we observe that the second vector is in the middle of the line because it goes 2, 6, 10 in the y component, so A is 4 so that it is in the middle of 1 and 7. We also get that B is 11.

In general one can solve for vectors given sufficient information.

Note that a quadratic equation with roots A and B can be written as $x^2-(A+B)x+AB$. Therefore, without solving the equation we can find the sum and product of the roots. As an example, given the equation x^2-6x+4 we know that the sum of the roots is 6 and the product of the roots is 4. We can even try to find the sum of squares of the roots, A^2+B^2. The trick is to try to write A^2+B^2 in terms of A+B and AB. A sensible thing would be to try $(A+B)^2$ as although it is not the answer (unfortunately) it is similar to the answer. In fact we get A^2+B^2+2AB but AB is known. In the end we just need to take the expression $(A+B)^2 - 2AB$. Therefore the sum of the squares of the roots of $x^2 - 6x + 4$ is equal to $6^2 - 2 * 4 = 28$. The product of the roots is $A^2B^2 = (AB)^2 = 16$. Therefore, without solving the equation, we deduce that the quadratic with roots A^2 and B^2 is $x^2 - 28x + 16$.

A quadratic with roots $\frac{1}{A}$ and $\frac{1}{B}$ has to have root sum $\frac{1}{A} + \frac{1}{B} = \frac{A+B}{AB}$ and root product $\frac{1}{AB}$. These are exactly $\frac{6}{4}$ and $\frac{1}{4}$ respectively so we get $x^2 - \frac{6}{4}x + \frac{1}{4}x = 0$, which (simplifying) is $4x^2 - 6x + 1 = 0$. Also, if we want to express $A^3 + B^3$ we have the useful identity $A^3 + B^3 = (A+B)^3 - 3AB(A+B)$ that we can check by algebra. Finally, if we want the difference between the roots, this is $|A - B|$ which is (if these are real) $\sqrt{(A-B)^2} = \sqrt{A^2 + B^2 - 2AB} = \sqrt{(A+B)^2 - 4AB}$.

There is a general theorem relating to this. Although you do not need to know this theorem for A level further maths or equivalent, it is the fundamental theorem of symmetric polynomials and more on it is in the “misc results” section of this website.

There is just one thing we need to be careful of. As an example, in the equation $5x^2 - 4x + 2$ we CANNOT say that the sum of the roots is 4. This is because that assumes the leading coefficient is 1. In reality the sum of the roots is $\frac{4}{5}$ and the product of the roots is $\frac{2}{5}$.

We will now look at some more graphs. Lets analyze what the graph $y = \frac{1}{x}$ might look like.

Observation 1: as x gets large $\frac{1}{x}$ goes to 0

Observation 2: If x is very small $\frac{1}{x}$ is very large

Observation 3: If x is negative then $\frac{1}{x}$ is negative and similarly it gets larger as x goes to 0 and smaller as x goes to infinity.

Observation 4: If $y = \frac{1}{x}$ then $x = \frac{1}{y}$ so we expect the graph to be symmetric about the line y=x.

We now will show an image of what the graph looks like and you will see it agrees with these observations.

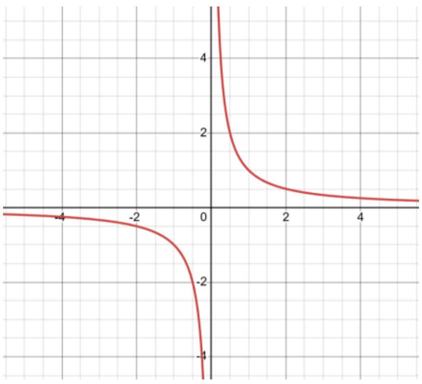


Image: Graph of $y = \frac{1}{x}$.

I’m gonna show another graph:

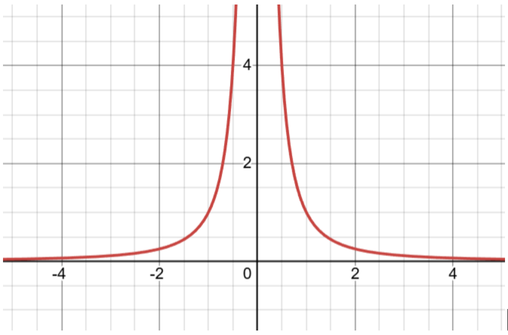


Image: Graph of $y = \frac{1}{x^2}$.

We will now do some examples of sketching graphs of cubic (degree 3) polynomials.

Example 1: Lets sketch the graph of $y = (x-1)(x+1)(x-2)$. This has roots at 1, -1 and 2. Therefore the graph must cross the x-axis there Note that if we expand this out, the x^3 term is largest. This goes to infinity as x goes to infinity and goes to negative infinity as x goes to negative infinity (since 3 is odd, if it were even it would go to positive infinity on both sides). Putting all of this together, here is what the graph looks like.

We note that by direct calculation, when x=0 y=2.

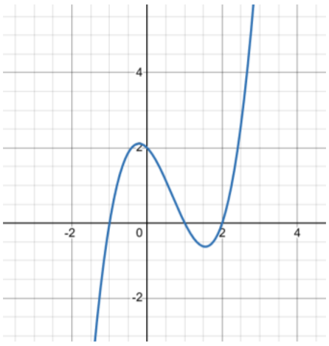
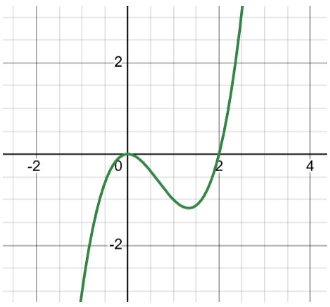


Image: Graph of $y = (x-1)(x+1)(x-2)$

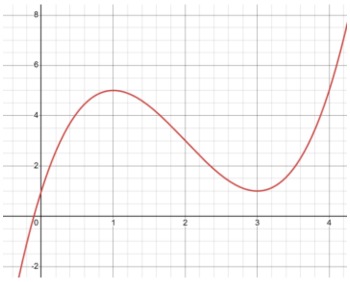
Another example: What if we have a "double root" at 0, ie $y = x^2(x-2)$, then what happens is that we cross the x-axis at 2, that is not a problem. Near x=0, what basically happens is that $x-2$ is about -2, so y is about $-2x^2$ so it looks like a downwardsparabola. Basically what happens is that in order for the graph to go to $-\infty$ and ∞ as x goes to $-\infty$ and ∞ respectively while being continuous only going through the x-axis 2 times, if you try to draw it yourself you will see that at some point you will need to hug the x-axis but not cross it. This always happens at a "double root" when we have a term squared in the factored polynomial. This may be confusing, but putting all of it together here is what the graph looks like:



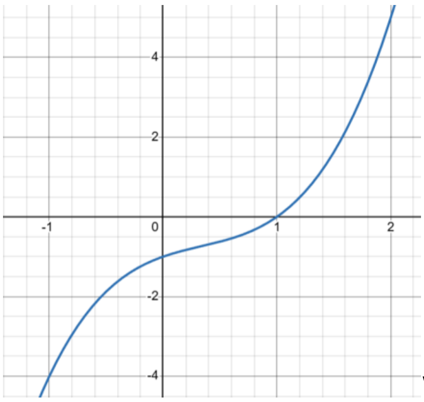
Lets try to sketch the graph of $x^3 - 6x^2 + 9x + 1$. In this case, we do not know how to factor it to find its roots, but we can find its "turning points" by solving for when the derivative is 0. The derivative is $3x^2 - 12x + 9$ which we can factor as $3(x^2 - 4x + 3) = 3(x-1)(x-3)$ so this is 0 when x=1 or x=3.

We therefore know that the graph is "flat" when x=1 or x=3. If $y = x^3 - 6x^2 + 9x + 1$ then by direct calculation, when x=1 y=5 and when x=3 y=1. Therefore what has to happen is as follows.

If x is large and negative, so is y. But x has to be increasing as it reaches (1,5) but cannot turn around until that point since we know all the points the derivative is 0, and its sign cannot jump since the derivative is a quadratic which is continuous. Now at this turning point it must reverse direction to reach (3,1), but then as x goes to infinity so does y so the graph must keep increasing. In the end, this is what it looks like:



Sometimes there may be no turning points. An example of this is the graph $x^3 - x^2 + x - 1$. In this case, we can spot that 1 is a root and that by direct calculation the y-intercept (the place where we cross the y-axis) is at (0,-1). If we differentiate and try to solve the quadratic we will see that there are no roots, but we can find a point of inflection – the point where the direction of curvature changes. This happens when the second derivative is 0, which happens when $6x - 2 = 0$, which is exactly when $x = \frac{1}{3}$. The second derivative is negative before that (so the graph should be curving downwards) until this point where it should start curving upwards. All together, here is what the graph looks like, remembering it must pass (0, -1) and (1, 0):



We see that when x is 1/3 the direction of curvature changes and all the properties described above are seen.

We can do the same tricks to sketch degree 4 polynomials, I will not go through this since it is the same. We can similarly do transformations on all these graphs.

Going back to reciprocal graphs, an asymptote is a line that the curve approaches and gets as close to as we want but never quite meets it. In the case $y = \frac{1}{x}$ the asymptotes are the lines $x = 0$ and $y = 0$.

In the case $y = \frac{3}{x-1} + 2$ we basically take that graph, shift it by 1 to the right, scale it up by 3 and add 2. The result of this is that the asymptotes move with the graph: the x=0 asymptote is a vertical line that moves to x=1 from shifting and the y=0 asymptote is a horizontal line that moves to x=2 from adding 2. All together, here is what the graph looks like:

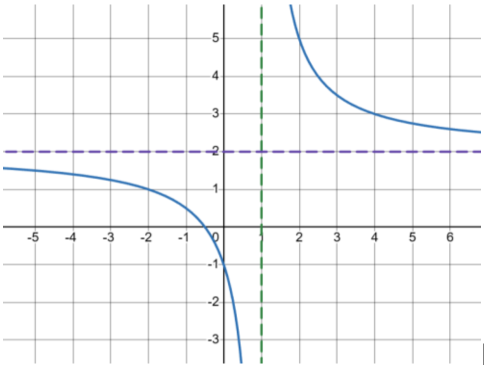


Image: Graph of $y = \frac{3}{x-1} + 2$

Given a function like $y = \frac{5x+3}{3x-7}$ we want to be able to i) graph it and ii) find its inverse. We can do this as follows.

First, notice that as x gets large it will approach $\frac{5}{3}$. We want to write it as $\frac{5}{3} + something$. Note that if we want our function to always (well, except at $x = \frac{7}{3}$) equal $\frac{5}{3}$ but have $3x - 7$ on the denominator then we can write $\frac{\frac{5}{3}(3x-7)}{3x-7}$ and notice the numerator (top of the fraction) will become 5x+something. In fact, in this case it will be $\frac{5x-\frac{35}{3}}{3x-7} = \frac{5}{3}$. Therefore to get to $\frac{5x+3}{3x-7}$ we just need to make the $-\frac{35}{3}$ become a 3 and the difference between these is $\frac{44}{3}$ so we just need to add $\frac{\frac{44}{3}}{3x-7}$ which is $\frac{44}{3(3x-7)}$. We can now rewrite $\frac{5x+3}{3x-7} = \frac{5}{3} + \frac{44}{3(3x-7)}$. This is useful because we now deduce that the asymptotes are at $x = \frac{7}{3}$ and $y = \frac{5}{3}$. Furthermore, we have isolated x so there is only 1 instance of x, so we can now try to find an inverse. I will just do it.

$$y = \frac{5}{3} + \frac{44}{3(3x-7)}$$

$$y - \frac{5}{3} = \frac{44}{3(3x-7)}$$

$$\frac{1}{\left(y - \frac{5}{3}\right)} = \frac{3(3x-7)}{44}$$

We want integers so we will multiply the numerator (top) and denominator (bottom) of the left fraction by 3 to get

$$\frac{3}{3y-5} = \frac{3(3x-7)}{44}$$

$$\frac{3 * 44}{3y-5} = 3(3x-7)$$

$$\frac{44}{3y-5} = 3x - 7$$

$$\frac{44}{3y-5} + 7 = 3x$$

$$\frac{44}{3(3y-5)} + \frac{7}{3} = x$$

We can now try to get a common denominator to simplify and write this as a single fraction.

$$\frac{44}{3(3y-5)}+\frac{7(3y-5)}{3(3y-5)}=x$$

$$\frac{44+7(3y-5)}{3(3y-5)}=x$$

$$\frac{21y+9}{9y-15}=x$$

$$\frac{7y+3}{3y-5}=x$$

This may seem complicated but you get used to it.

Note that by pythagoras the distance of any point in the plane to the origin is given by $\sqrt{x^2+y^2}$. Therefore any curve described by $x^2+y^2=constant$ is the set of points a constant distance from the origin and is therefore a circle. A circle with radius r centered at the origin is given by $x^2+y^2=r^2$. Similarly, a circle of radius r and center (a,b) is given by $(x-a)^2+(y-b)^2=r^2$.

Example: The equation of a circle is $x^2+y^2+4x-6y+4=0$ and we want to find its center and radius. We do this by completing the square on x^2+4x and y^2-6y. In the end, we get

$(x+2)^2-4+(y-3)^2-9+4=0$ so $(x+2)^2+(y-3)^2=9$. Therefore the center is at (-2, 3) and the radius is $\sqrt{9}$ which is 3.

A radian is a unit of measuring angle with the property that π radians is 180 degrees. This is useful because it makes it so that if I have a circle of radius r and I walk r units around its circumference I cover a distance of 1 radian. We will generally use radians from now on.

Now suppose we have a sector of a circle.

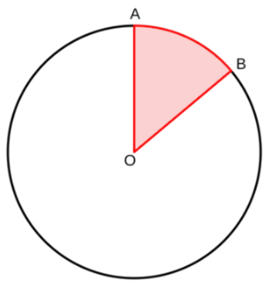
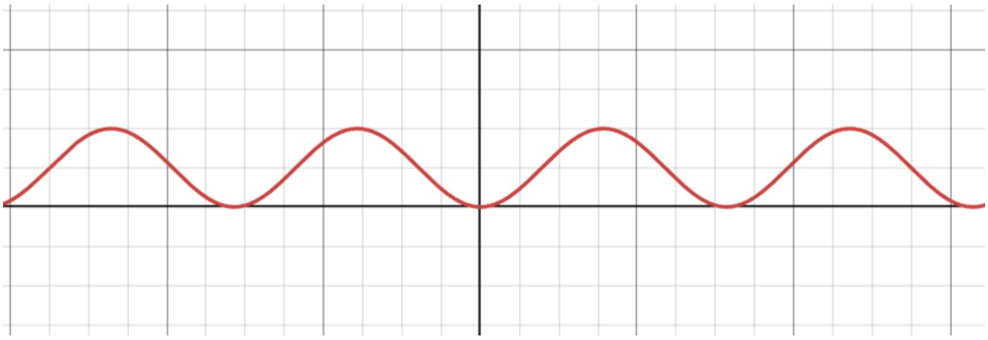


Image of a sector of a circle.

Now suppose that angle AOB is k radians, then we want to find the area of a sector. It should be obvious to you that the area is proportional to k, and we know that if k is 2π the area is πr^2. Therefore the area of the sector is given by $\frac{kr^2}{2}$.

Now recall from the last level what a graph of sin(x) or cos(x) looks like.

Now I will show below an image of the graph of $y=(\sin(x))^2$, which mathematicians normally write as $\sin^2(x)$ so I will do that from now on.



Interesting, it looks exactly like a sine wave.

In fact, it looks exactly like a cosine wave with the following elementary transformations: It looks as though we can say $\sin^2(x) = \frac{1+\cos(2x)}{2}$. This turns out to be a special case of two identities which I will prove in level 4 and for now we will assume them to be true.

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

I will state two limits which apply when we are working in radians which we also prove in level 4 using a pretty geometric argument:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$$

Now using these identities I will derive a lot of properties.

$\sin(A-B) = \sin\big(A + (-B)\big)$. Using the following properties (known by symmetries of the graph): $\cos(-x) = \cos(x)$ and $-\sin(x) = \sin(-x)$, we get the following formulae:

$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

Now we can find the following formulae:

$$\sin(2A) = \sin(A+A) = \sin(A)\cos(A) + \cos(A)\sin(A) = 2\sin(A)\cos(A)$$

$$\cos(2A) = \cos(A+A) = \cos(A)\cos(A) - \sin(A)\sin(A) = \sin^2(A) - \cos^2(A)$$

It turns out we can simplify that last formula even further.

Consider a right angled triangle with hypotenuse of length 1 and an angle of length A, then by pythagoras we get the identity $\sin^2(A) + \cos^2(A) = 1$. Using this, we can write the following:

$$\cos(2A) = 2\cos^2(A) - 1$$

$$\cos(2A) = 1 - 2\sin^2(A)$$

Note that if we again consider a right angled triangle and use the notation H is hypotenuse (long side), A is the side adjacent to the angle x, and O is the side opposite to the angle x, we get the following result: $\tan(x) = \frac{O}{A} = \frac{O/H}{A/H} = \frac{\sin(x)}{\cos(x)}$.

Now we do some trickery to get a formula for $\tan(A + B)$. We write

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin(A)\cos(B) + \cos(A)\sin(B)}{\cos(A)\cos(B) - \sin(A)\sin(B)} = \frac{\frac{\sin(A)\cos(B)}{\cos(A)\cos(B)} + \frac{\cos(A)\sin(B)}{\cos(A)\cos(B)}}{\frac{\cos(A)\cos(B)}{\cos(A)\cos(B)} - \frac{\sin(A)\sin(B)}{\cos(A)\cos(B)}}$$

$$= \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$$

Similarly (Using tan(-x)=-tan(x) by symmetry of the graph),

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

Now using the addition formulae and the limits above we will try to differentiate sin(x) in radians.

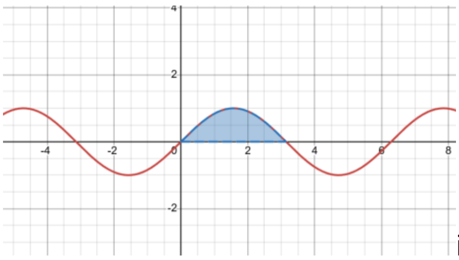
$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0}\frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$
$$= \lim_{h\to 0}\frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} = \sin(x)\lim_{h\to 0}\frac{\cos(h) - 1}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$
$$= \cos(x)$$

So the derivative of sin is cos. Isn't that elegant.

Also, differentiating sin effectively shifts the graph $\frac{\pi}{2}$ radians to the left. We can therefore reason that if we differentiate again the same thing will happen. We get the result

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

Example: By integration, if we plot the sine graph in radians then we can find what the area is under one of the bumps,



ie, the area in this image.

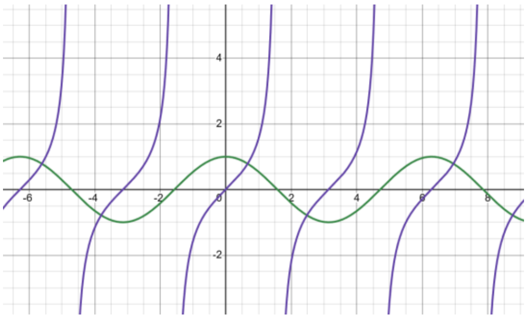
We note that since we are in radians, then as suggested by the graph, sin is 0 at exactly all multiples of pi. We note that an antiderivative of sin is -cos.

$$\int_0^{\pi}\sin(x)\,dx = (-\cos(\pi)) - (-\cos(0))$$

Now doing the calculation (possibly converting to degrees in your head if you're not comfortable with radians yet), we get 2. So the area is exactly 2 units squared. This is pretty cool.

Example:

Lets solve the equation $\cos(x) = \tan(x)$, ie these points of intersection:



It is good to know how to solve equations using these trigonometric identities and determine how many solutions exist on any interval, and so we will walk through this example.

We simplify by writing cos(x) as c, sin(x) as s and tan(x) as t. The identities we know are $c^2 + s^2 = 1$ and $t = \frac{s}{c}$. We write

$$c = t$$

$$c = \frac{s}{c}$$

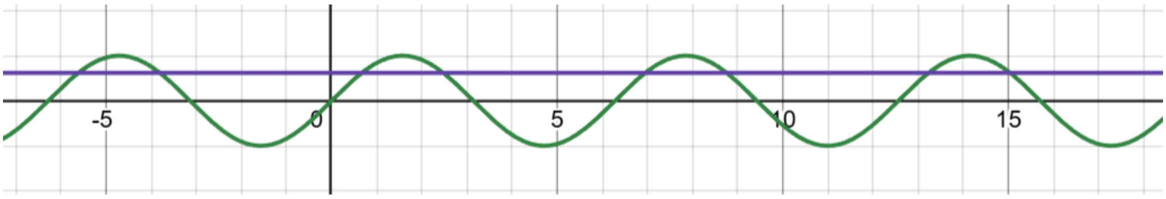
$$c^2 = s$$

$$1 - s^2 = s$$

The solution to this quadratic equation in s is $\frac{-1 \pm \sqrt{5}}{2}$ by the quadratic formula.

We therefore get $\sin(\mathrm{x}) = \frac{-1 \pm \sqrt{5}}{2}$ as solutions, but recall from the previous level that we can't really invert sin even though it is tempting to just do arcsin of both sides.

What happens is if $\sin(x) = c$ then $x = \arcsin(c)$ is one solution, and we get a symmetric solution by taking $\frac{\pi}{2} - \arcsin(c)$ and then more solutions by adding multiples of 2π.



This image shows what is going on graphically.

Note that since for real numbers, sin is always between -1 and 1, we really only have $\sin(\mathrm{x}) = \frac{\sqrt{5}-1}{2}$ as valid solutions.

Note that we are effectively looking for exactly the intersection points in the image above as the value of x plotted was $\frac{\sqrt{5}-1}{2}$ which is about 0.618.

Now we note that there is a wave every 2π. And 2 intersection points on each.

We can therefore deduce that, for example, between 0 and 20π there are 20 solutions.

We can now introduce some new definitions:

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\cot(x) = \frac{1}{\tan(x)}$$

As usual this is not defined whenever the denominators are 0. We define cot to be 0 whenever tan is undefined.

We get some new useful identities. If we start with $\sin^2(x) + \cos^2(x) = 1$ and divide both sides through by $\cos^2(x)$ we get the useful identity $\tan^2(x) + 1 = sec^2(x)$ at every point where these things are defined.

We similarly get the identity $\cot^2(x) + 1 = csc^2(x)$ if we divide the pythagorean identity by $\sin^2(x)$ instead of $\cos^2(x)$.

Example: Lets investigate how many ways there are to rearrange the letters in the word "Maths". There are 5 things to put in the first position and after that there are 4 things to put in the second position and so on. We end up with 5*4*3*2*1=5!=120 total ways.

Lets investigate how many ways there are to rearrange the letters in the word "Skibidi". We would want to say there are 7! but then for each way there are 6 ways to reorder the 3 i's and each of these are identical so we have to divide by 6. In the end we have $\frac{7!}{6}$ which is 840 ways.

Lets investigate how many ways there are to arrange the 3 i's in the word "Skibidi" if we don't care how the other letters are arranged. For each rearrangement (or "permutation") there are 4!=24 ways to arrange the other letters so we have to divide by 4!. We end up with 35 ways.

In general, the number of ways to choose r things from n things is "n choose r" or $\binom{n}{r}$ which is equal to the quantity $\frac{n!}{r!(n-r)!}$. These are called binomial coefficients.

We will now prove a useful identity, which says that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$. It is certainly possible to do this from the factorial definition but there is a much nicer way. We consider the fact that we either pick the last element or we don't, and in these cases we respectively have to pick r-1 elements from the remaining n-1 or r elements from the remaining n-1.

We now consider how this applies to expanding $(1 + x)^n$.

Example: If we expand $(1 + x)^5 = (1 + x)(1 + x)(1 + x)(1 + x)(1 + x)$ then consider what, say, the final coefficient of x^2 will be. This is the number of ways to pick 2 x's from the 5. This is exactly equal to $\binom{5}{2} = \frac{5!}{2!3!} = \frac{120}{2*6} = 10$. So 10 is the coefficient of x^2 when we expand this out.

We write (Binomial theorem for integers) that $(1 + x)^n = \sum_{r=0}^{n} x^r \binom{n}{r}$.

Example: Suppose that the x^3 coefficient in $(1 + ax)^6$ is 540. Find the value of a given that a is some real number.

What we do here is note that the x^3 term is $\binom{6}{3}(ax)^3$ so we want to solve $\binom{6}{3}a^3 = 540$. It turns out that $\binom{6}{3}$ is 20 so we want $a^3 = 27$. Therefore a=3 (We can cancel cubes since on the real numbers taking something to the third power is a one-to-one function since 3 is odd).

We define $\binom{n}{r}$ to be 0 if r is a negative integer or a positive integer greater than n.

Note that we can take infinite sums like $\sum_{n=1}^{\infty} f(n)$. For example, $\sum_{n=1}^{\infty} \frac{1}{10^n}$ which is just 0.1+0.01+0.001 and so on. We define an infinite sum as the limit of the finite sums (or partial sums) when this limit exists. In the example above we get 0.111111... with infinitely many 1's, or $\frac{1}{9}$.

This limit does not always exist, $\sum_{n=1}^{\infty} n$ is 1+2+3+4+... which clearly goes to infinity, it clearly does not go to any finite number, surely not $-\frac{1}{12}$ for example.

The more surprising thing is that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ slowly drifts off to infinity. The proof is quite elegant but you need to go see level 7 numbers and sets.

Another surprising thing is that you can't always reorder terms in an infinite series and get the same result, so you need to be careful. See level 4 for further details.

Definition: A geometric sequence is a sequence of the form $a, ar, ar^2, ar^3, \ldots$ An example is 1, 2, 4, 8, …

A geometric series when we add them together like $a + ar + ar^2 + ar^3 + \cdots + ar^n$ is given by the formula $a * \frac{1-r^{n+1}}{1-r}$. The reason is because if we expand $a(1-r)(1 + r + r^2 + \cdots + r^n)$ factoring out the a we see that almost everything cancels and we get $1 - r^{n+1}$. Something that is not hard to justify but we do it in level 4 is that if |r|<1 (ie r between -1 and 1) then the sum of the infinite geometric series is $\frac{a}{1-r}$ (Hint about why: What happens as n gets large in the various cases?).

Definitions: A sequence is periodic if it has a repeating pattern like 1,4,2,1,4,2,1,4,2,...

It is eventually periodic if it does stuff first and then becomes periodic, such as 10,5,16,8,4,2,1,4,2,1,4,2,...

A sequence converges if it gets closer to a value and eventually gets and stays as close as we want that value. For example, 2.1, 2.01, 2.001,... converges to 2, but does not necessarily converge to 1 even though it gets closer to 1, since 2 is the only value it gets arbitrarily close to. It does not approach some weird value like 2.000...001 because two real numbers are only different if you can zoom into the number line a finite amount and then they are distinguished.

A sequence is divergent if it does not converge, but this does not necessarily mean it goes to infinity – it could be something like 0, 1, 0, 1, 0, 1, ...

We now state the generalized binomial theorem, and we prove this in level 4 since the proof is a bit technical.

If r is some real number, then $(1+x)^r$ is given by the following infinite sum valid when |x|<1:

$$1 + rx + \frac{r(r-1)}{2}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \frac{r(r-1)(r-2)(r-3)}{4!}x^4 + \cdots$$

Note that if r is a positive integer this still holds but then what happens is that after the r'th term we have a term like (r-r) so all terms collapse to 0.

Lets do an example. The series for $\frac{1}{1+x} = (1+x)^{-1}$ is given by $1 - x + x^2 - x^3 + x^4 + \cdots$ when |x|<1 which is exactly what the geometric series we just did would predict.

The series for $\sqrt{1+x} = (1+x)^{\frac{1}{2}}$ is given by $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$

We can expand $(a+bx)^r$ as follows:

Write $(a+bx)^r = a^r\left(1 + \left(\frac{b}{a}\right)x\right)^r$. Here we expand in $\left(\frac{b}{a}\right)x$ valid when $\left|\left(\frac{b}{a}\right)x\right| < 1$ so $|x| < \left|\frac{a}{b}\right|$.

Example:

Lets suppose we want to find the square root of 3 accurately. You may not necessarily come up with this, but I will write $\sqrt{1.08} = \sqrt{0.6^2 * 3} = 0.6\sqrt{3}$. Now expand $\sqrt{1+x}$ with x=0.08. We are making this small because we generally have better approximations the smaller x is since the terms in higher powers of x get smaller quicker. Expanding this up to the term in x^2 gives

$$1 + \frac{1}{2}(0.08) - \frac{1}{8}(0.08)^2 = 1.0392$$

Dividing by 0.6 gives 1.732. This is pretty good considering the actual value of the square root of 3 is about 1.7320508. So the generalized binomial theorem is useful for this kind of application.