Contents

Note: The reason these notes are not self contained in the context of notes for the Cambridge course is because I didn't make notes for physical parts due to the fact that I like maths and not physics. However, I rigorously prove things that they don't do in lectures, at least I try my best.

1 Introduction

We will generally do things in $\mathbb{R}^3$ to get intuition, many but not all of the results of this course will be valid in higher dimensions. This course turns out to be incredibly difficult to formalize even compared to differential equations but I'm doing my best.

Of course, to differentiate a vector we differentiate each of the components. It therefore follows from the one variable product rule that

$$\frac{d}{dx}(a.b) = \dot{a}.b + a.\dot{b}$$

$$\frac{d}{dx}(a \times b) = \dot{a} \times b + a \times \dot{b}$$

Where in the second expression we need to be careful to write a and b in the right order.

We define the integral of a vector component-wise, and we have

$$\int_{x_1}^{x_2} \dot{a}(x)\, dx = a(x_2) - a(x_1)$$

Note that we write $\frac{\partial}{\partial x_i}$ as ∂_i or D_i. And $(Df)_{ij} = D_i f_j$.

The notation of vector calculus is annoying and often you need to infer the meaning, unfortunately.

We can write the multivariate chain rule in summation convention as

$$\frac{\partial f}{\partial u_i} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial u_i}$$

For analytic (meaning having a local taylor series) vectors v=(x,y,z) and a small change h

$$f\left(v+h\right)=f\left(x,y,z\right)+\frac{\partial f}{\partial x}h_x+\frac{\partial f}{\partial y}h_y+\frac{\partial f}{\partial z}h_z+\frac{1}{2}\left(h_x\frac{\partial}{\partial x}+h_y\frac{\partial}{\partial y}+h_z\frac{\partial}{\partial z}\right)^2 f+\dots$$

We think of symbols like df and dx as follows: If we write $df=f'\left(x\right)dx$ we mean that this holds after we integrate them. No "infinitesimal" nonsense that the lecturer talks about.

These have rules like (for vectors f) $df=\frac{\partial f}{\partial x_i}dx_i$. Honestly its so much worse to say "summation convention" than to just write sigma but whatever.

Definition. (Dual numbers) We introduct this to talk about infinitesimals like "dx" rigorously. Just like we can define imaginary numbers such that $i^2=-1$ we will define ε such that $\varepsilon\neq 0,\ \varepsilon^2=0$. This seems wrong but so did the imaginary numbers, this is actually a consistent system where we can write numbers as $a+b\varepsilon$.

In this framework, let f be an analytic function at x_0, then df means $f\left(x_0+\varepsilon\right)-f(x_0)$ which by a taylor expansion and the definition of ε is just $\varepsilon\frac{df}{dx}$. We can now say $d\left(fg\right)=g\left(df\right)+f(dg)$ for example, and $df=\frac{\partial f}{\partial x_i}dx_i$ for multivariate functions (Simply add ε_i to the i'th component where each one is a different number with the $\varepsilon_i^2=0$ property, mathematicians really do invent new numbers to do stuff).

Proposition. This is consistent with integration notation, ie $\int_{x=x_0}^t f\left(x\right)dg(x)=\int_{x=x_0}^t f\left(x\right)\frac{dg}{dx}dx$ as long as g and f have continuous derivatives in this range.

Proof. (Level 4 revision) By the chain rule the above expressions both have the same derivatives wrt x (for the left differentiate with respect to g then multiply by $\frac{dg}{dx}$).

□

A differential operator like $\nabla=\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}$ is a function that sends vectors to other vectors. Or we can write it compactly as $e_i\frac{\partial}{\partial x_i}$

Definition. A scalar or vector field is a set of scalars or vectors that depend on position. Often in this course, when someone says "scalar field" or "vector field", it is implied that it is infinitely differentiable.

Example: If we have a scalar field $f\left(v\right)=\left|v\right|^2=x^2+y^2+z^2$ then $\nabla f=(2x,2y,2z)=2v$

[Lecture 1 ends]

Example. $\nabla\left(a.v\right)=\nabla\left(ax+by+cz\right)=\left(a,b,c\right)=a$ where $a=\left(a,b,c\right),\ v=\left(x,y,z\right)$.

Example. If $f\left(v\right)=x+y^2+z^4$ then $\nabla f\left(a\right)=\nabla f(b)$ if and only if a-b is parallel to the x-axis.

Proof. This is true if and only if $\left(1,2a_2,4a_3^3\right)=\left(1,2b_2,4b_3^3\right)$ are equal which happens if and only if the y and z coordinates are equal.

□

We need to be careful – for example $\nabla\left(f\left(2x\right)\right)\neq\left(\nabla f\right)\left(2x\right)$. Notation for these things can be ambiguous and you may sometimes need to use context. This is not my fault or the lecturer's, rather it is the fault of hundreds of years of mathematicians.

We can write the chain rule as $\frac{d}{dt}f\left(v\left(t\right)\right)=\dot{v}.\nabla f$. Grad may be different depending on the basis, if we use the standard basis then ∇ is shorthand for ∇_e but we can write ∇_x for any basis x.

Definition. (Divergence)

Let F be a vector field $\mathbb{R}^n \to \mathbb{R}^n$, then $div(F) = \sum_{i=1}^{n} \frac{\partial F_i}{\partial e_i}$. We can write this as $\nabla.f$ where $\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$. This is a notational trick and not an actual dot product between numbers.

There is an intuition for divergence but the argument for why will have to wait until later. We will see that if some fluid follows the vector field the divergence measures the rate at which it flows in or out of a region.

Example. If $f(v) = v$ then $div(f) = x + y + z$. If $f(x) = (-y,\ x,\ 0)$ then $div(f) = 0$.

Definition. (Curl) We write as notation

$$Curl\,(f) = \nabla \times f$$

Ie, in 3 dimensions,

$$Curl\,(f) = \begin{pmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix}$$

$$Curl(f)_i = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j}$$

$$Curl(f) = e_i \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j}$$

There is an intuition for this which we will also see later.

Example: If A is a 3x3 symmetric matrix then $Curl\,(Ax) = e_i \varepsilon_{ijk} \frac{\partial A_{kl} x_l}{\partial x_j} = e_i \varepsilon_{ijk} A_{kj} = 0$

The last part follows from A being symmetric.

Also, if A is any matrix, div(Ax)=Tr(a), this is easy to check.

[Lecture 2 ends]

Grad is something where you take a scalar and get a vector. Div is something where you take a vector and get a scalar. Curl is something where you take a vector and get a vector.

If we have a vector u then the derivative in the direction of u is $u.\nabla = u_i \frac{\partial}{\partial x_i}$. For a fixed vector u It is true that $(u.\nabla)\, f = u.\,(\nabla f)$, but you have to be careful as this kind of thing is not always true. We can get a vector by applying this operator to a vector field. If f is a vector then $(u.\nabla)\, f = u.\,(\nabla f)$ is false as the right handside does not make sense due to trying to do grad of a vector.

Example. Suppose $u = \begin{pmatrix} 0 \\ x \\ y \end{pmatrix}$, $F = \begin{pmatrix} yz \\ 0 \\ x \end{pmatrix}$, then $(u.\nabla)\, F = \left(x \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \right) F = x \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y^2 + xz \\ 0 \\ 0 \end{pmatrix}$

By convention we consider u to be a unit vector when we compute the directional derivative.

Definition. A level surface is the higher dimensional analog of a contour plot, ie something like $f(x, y) = 0$ or $f(x, y, z) = 0$.

Proposition. Suppose f is a function $\mathbb{R}^n \to \mathbb{R}$ grad(f) is not 0 at some point and continuous there, then the level surface is continuously differentiable in some neighborhood around that point – This is a generalization of something which we proved in level 6 about implicit functions/implicit differnetiation.

Proof. Write a point in $\mathbb{R}^n$ as (x,y) with x in $\mathbb{R}^{n-1}$. After possibly reordering coordinates suppose that $f_y \neq 0$. Use the implicit function theorem as stated in the analysis lemmas document with "n"=n-1 and "k"=1. Then the result follows.

□

Proposition. Suppose f is a function $\mathbb{R}^n \to \mathbb{R}$ grad(f) is not 0 at some point and continuous there, then the level surface in $\mathbb{R}^n$ is perpendicular to grad(f).

Proof. Since f is constant along the level surface, if means that if u is any direction tangent to the level surface, then the directional derivative $u.\nabla f$ is 0 since the directional derivative is 0 as f is constant. Since this is true for all tangent directions, the result follows.

□

Example. The direction of $|x|^2$'s largest rate of change (which is grad, see differential equations) points outwards from the origin, and its level surfaces are spheres and it is clear that these are perpendicular.

Grad can be thought of as either the slope direction of the graph of the function that is steepest uphill, or the direction perpendicular to the contour diagram.

We can prove some identities using suffix notation (summation convention).

Suppose f is a scalar field and F a vector field,

$$\nabla.(fF) = \frac{\partial}{\partial x_i}(fF_i) = \frac{\partial f}{\partial x_i}(F_i) + f\frac{\partial F_i}{\partial x_i} = \nabla f.F + f\nabla.F$$

$$(F \times (\nabla \times G))_i = \varepsilon_{ijk}F_j(\nabla \times G)_k = \varepsilon_{ijk}F_j\varepsilon_{klm}\frac{\partial}{\partial x_l}G_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})F_j\frac{\partial G_m}{\partial x_l} = F_j\left(\frac{\partial G_j}{\partial x_i} - \frac{\partial G_i}{\partial x_j}\right)$$

Doing the same thing for $(G \times (\nabla \times F))_i$ and adding them together gives

$$F_j\left(\frac{\partial G_j}{\partial x_i} - \frac{\partial G_i}{\partial x_j}\right) + G_j\left(\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j}\right)$$

Reversing the product rule for 2 of the terms and writing the others in terms of div we get the following:

$$(G \times (\nabla \times F))_i + (F \times (\nabla \times G))_i = \frac{\partial F_j G_j}{\partial x_i} - ((F.\nabla)G)_i - ((G.\nabla)F)_i$$

We now get

$$\nabla(F.G) = (F \times (\nabla \times G)) + (G \times (\nabla \times F)) + ((F.\nabla)G) + ((G.\nabla)F)$$

By considering components.

Formulae involving dot and cross products DO NOT WORK if we have a ∇. It is just a NOTATIONAL TRICK.

If r is the radius, then by differentiating the square root (I won't go through this, but the idea is to use the fact that $\frac{d}{dx}\left(\sqrt{x^2+c}\right) = \frac{x}{\sqrt{x^2+c}}$),

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

[Lecture 3 ends]

Definition. The laplacian is $\nabla^2 f = \nabla.(\nabla f) = div\,(grad\,(f)) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial x_i \partial x_i}$

We sometimes write this as a right side up triangle.

Example. If $f(x, y, z) = ax^2 + by^2 + cz^2$ then $\nabla^2 f = 2a + 2b + 2c$.

Example. $\nabla^2|x|^2 = \nabla^2\left(x^2 + y^2 + z^2\right) = 6$

Example. Let $f(x) = (a.x)^2$. We first need to find the gradient.

By the chain rule, $\nabla\left((a.x)^2\right) = 2(a.x)\nabla(a.x)$. Now $\nabla(a.x) = a$ so we need $Div(2a(a.x))$. Using the identity from last lecture $\nabla.(fF) = \nabla f.F + f\nabla.F$ we get that this is $2(a.x)\,div\,(a) + 2a.grad(a.x)$ and a is constant so div(a) is 0 so in the end we get 2a.a which is $2|a|^2$.

Functions that obey $\nabla^2 f = 0$ are called harmonic functions. These are important in both pure maths and applied maths.

Suppose f has rotational symmetry, such as only depending on the radius. The multivariate normal is an important example of this.

$$\nabla^2 f(|v|) = \nabla.(\nabla f(|v|)) = \nabla.(f'(|v|)\nabla|v|)$$

Now the i'th component of the inside is= $\nabla.\left(f'(|v|)\frac{x_i}{|v|}\right)$ so the divergence is $\frac{\partial}{\partial x_i}\left(f'(|v|)\frac{x_i}{|v|}\right)$. We now use the ordinary product and quotient rules to get

$$\frac{\partial}{\partial x_i}\left(f'(|v|)\frac{x_i}{|v|}\right)$$

$$= \frac{x_i}{|v|}\frac{\partial}{\partial x_i}(f'(|v|)) + f'(|v|)\frac{\partial}{\partial x_i}\left(\frac{x_i}{|v|}\right)$$

$$= \frac{x_i}{|v|}f''(|v|)\frac{\partial|v|}{\partial x_i} + f'(|v|)\frac{|v|\delta_{ii} - x_i\frac{\partial|v|}{\partial x_i}}{|v|^2}$$

$$= \frac{x_i}{|v|}f''(|v|)\frac{x_i}{|v|} + f'(|v|)\frac{|v|\delta_{ii} - \frac{x_i x_i}{|v|}}{|v|^2}$$

Note that by pythagoras $x_i x_i = |v|^2$. Therefore in 3 dimensions, we end up with

$$= f''(|v|) + 2\frac{f'(|v|)}{|v|}$$

Where this "2" is generally the dimension minus 1.

For a vector field F(x) we define $\nabla^2 F$ as having $\nabla^2 F_i$ in the i'th component.

$$(\nabla\times(\nabla\times F))_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{klm}\frac{\partial F_m}{\partial x_l} = \frac{\partial^2 F_j}{\partial x_i\partial x_j} - \frac{\partial^2 F_i}{\partial x_j\partial x_j} = \frac{\partial}{\partial x_i}(\nabla.F) - \nabla^2 F_i$$

Therefore we can write $\nabla^2 F$ as $\nabla(\nabla.F) - \nabla\times(\nabla\times F)$.

We say a vector field is conservative if it is the grad of some single valued scalar field f. I emphasise single valued because it means you cannot integrate it around a path and not get back to where you started but in general there is a single valued antiderivative.

A conservative field F normally has 0 curl (it always does if F is continuously differentiable) because $\nabla\times(\mathrm{F}) = \nabla\times(\nabla \mathrm{f}) = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k}$. If F is continuously differentiable then f is 2x continuously differentiable and then by symmetry of mixed partials the term above ends up going to 0. Later in the course we will prove a partial converse (That if F has 0 curl on a simply connected domain it is conservative, a counterexample is $\frac{1}{x^2+y^2}$ where the antiderivative $\arctan\left(\frac{y}{x}\right)$ is not single valued, it picks up a factor of 2π if you wrap around the origin).

We note that in dimensions higher than 2, we define simply connected to mean any closed curve can be shrunk to a point. This means that it can have holes, since any loop in $\mathbb{R}^3\backslash 0$ can be shrunk to a point even though it has holes.

[Lecture 4 ends]

Definition. We say a vector field F is solenoidal if the divergence is 0 everywhere.

Suppose there exists a twice continuously differentiable vector field A such that $F = curl(A) = \nabla\times A$, then

$$\nabla.(\nabla\times A) = \frac{\partial}{\partial x_i}\varepsilon_{ijk}\frac{\partial A_k}{\partial x_j} = 0$$

(again by symmetry of mixed partials since A is twice continuously differentiable).

Definition. A curvilinear coordinate system in 3D is a parametrization (u, v, w) of $\mathbb{R}^3$ that is surjective.

Let x be the map defined by the differentiable parametrization $(u, v, w) \to \mathbb{R}^3$, then we define

$$h_u = \left|\frac{\partial x}{\partial u}\right|, \; h_v = \left|\frac{\partial x}{\partial v}\right|, \; h_w = \left|\frac{\partial x}{\partial w}\right|$$

We define e_u as $\frac{1}{h_u}\frac{\partial x}{\partial u}$ and similarly for $e_v, \; e_w$. These are now unit vectors.

Now the chain rule and using the dual number system implies (whenever everything is analytic)

$$dx = h_u du e_u + h_v dv e_v + h_w dw e_w$$

By "dividing" by dt and using the chain rule we see that these can be interchanged in a line integral provided that the map $x \to u, v, w$ has a continuous derivative which it clearly does for spherical and cylindrical polar coordinates (We will see what a line integral is later).

We cannot say the position vector is $ue_u + ve_v + we_w$ unless we are in the case that x is a linear transformation that sends unit basis vectors to unit vectors.

Suppose a displacement keeps u fixed but lets v and w vary. Then as described in earlier lectures, we have that this is perpendicular to e_u.

2 Coordinate systems

We define cylindrical polar coordinates as $x = p\cos(\phi), \; y = p\sin(\phi), \; z = z$. We can write this as

$$p = \sqrt{x^2 + y^2}, \; z = z$$

The one for ϕ is a bit harder, we want to say $\phi = \arctan\left(\frac{y}{x}\right)$ but this cannot be made continuous and single valued and defined everywhere. So we will not worry about inverting this.

Surfaces of constant ϕ are vertical planes. Surfaces of constant p are cylinders. Surfaces of constant z are horizontal planes. These all intersect perpendicularly, but anyway it is easy to see that in this case we have e_ϕ, e_p, e_z is orthonormal everywhere.

If we compute it we get

$$e_p = (\cos(\phi), \; \sin(\phi), \; 0), \; e_\phi = (-\sin(\phi), \cos(\phi), 0), \; e_z = (0, 0, 1)$$

Which is indeed orthonormal.

Now from the definitions, $x = pe_p + ze_z$ where x is any 3D vector. (There is no ϕe_ϕ).

Also, $dx = dpe_p + pd\phi e_\phi + dze_z$.

We will define spherical polar coordinates as follows:

$$x = r\sin\theta\cos\phi, \; y = r\sin\theta\sin\phi, \; z = r\cos\theta$$

Figure 1 shows what is going on.

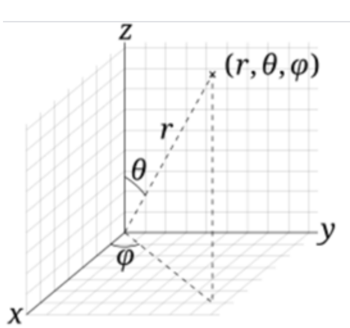


Figure 1

We can write

$$r = \sqrt{x^2 + y^2 + z^2}$$

We can pretend to write

$$\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

$$\phi = \arctan\left(\frac{y}{x}\right)$$

Here we restrict θ to be between 0 and π.

We now get the following orthonormal vectors:

$$e_r = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix},\ e_\theta = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix},\ e_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

We note that $x = re_r$. Also, we say $dx = dre_r + rd\theta e_\theta + r\sin\theta d\phi e_\phi$, but of course this is dodgey as we haven't defined d(anything).

And we note that $h_r = |x_r| = 1,\ h_\theta = r,\ h_\phi = r\sin(\theta)$

As usual with this course, you need to be careful as other resources might use slightly different symbols in different orders.

The e's are not always orthonormal but they are in these examples. They also depend on the point we are at and are not constant vectors.

In suffix notation we write

$$\nabla f.e_u = \frac{\partial f}{\partial x_i}\frac{1}{h_u}\frac{\partial x_i}{\partial u} = \frac{1}{h_u}\frac{\partial f}{\partial u}$$

by the chain rule in an orthonormal coordinate system and the fact that dot products therefore act as projections. This lets us express ∇f in terms of u, v and w.

[Lecture 5 ends]

Continuing where we left off writing ∇f in terms of our new coordinate system, we write

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}e_u + \frac{1}{h_v}\frac{\partial f}{\partial u}e_v + \frac{1}{h_w}\frac{\partial f}{\partial u}e_w$$

For example, in cylindrical polar coordinates,

$$\nabla f = \frac{\partial f}{\partial p}e_p + \frac{1}{p}\frac{\partial f}{\partial \phi}e_\phi + \frac{\partial f}{\partial z}e_z$$

In 2D polar coordinates just ignore z. In spherical polar coordinates,

$$\nabla f = \frac{\partial f}{\partial r}e_r + \frac{1}{r}\frac{\partial f}{\partial \theta}e_\theta + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \phi}e_\phi$$

Example. Let $f(v) = |v|^2$. We derived earlier that $\nabla f = 2v$ and will now try to do this again.

In cylindricals, from earlier we have $x = pe_p + ze_z$ so $f(p, \phi, z) = p^2 + z^2$ and therefore $F_p = p$ and $F_\phi = 0$ and $F_z = z$. Here when we take partial derivatives we are taking the other two variables to be constant and using the fact that this is orthonormal to help us.

From the formula$\nabla f = \frac{\partial f}{\partial p}e_p + \frac{1}{p}\frac{\partial f}{\partial \phi}e_\phi + \frac{\partial f}{\partial z}e_z$ we get $2pe_p + 2ze_z = 2v$ as required.

Recall that in sphericals, $v = re_r$ so $f(r, \theta, \phi) = r^2$, $f_r = 2r$ and $\nabla f = 2re_r = 2v$.

Now let F(v)=v and we want to calculate Div(F). We want to find div, curl and the laplacian (div of grad) of stuff in other orthogonal coordinate systems. A way we might try to do this would be to say that our new basis is orthonormal so we just do the dot product trick, but we have to be very careful. We might write

$$F = re_r$$

$$\nabla = \frac{\partial}{\partial r}e_r + \frac{1}{r}\frac{\partial}{\partial \theta}e_\theta + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi}e_\phi$$

$$\nabla.\mathrm{F} = \frac{\partial r}{\partial r} + \frac{1}{r}\frac{\partial 0}{\partial \theta} + \frac{1}{r\sin(\theta)}\frac{\partial 0}{\partial \phi}$$

However, there is a problem with this approach. The problem is that it is a notational trick that we have not justified outside of the world of constant orthonormal vectors and we cannot assume it to work. In these other coordinates we also have to do the rescaling by h factors and generally it is a mess. We see that in this case it gives us the wrong answer – 1 and not 3. Instead, we will develop some integral theorems then come back to this. Note that here we will always be working in orthogonal coordinate systems and assume we are at points where everything is infinitely differentiable, or at least differentiable some number of times for everything to be justified that I'm too lazy to work out so I'll just say infinity to be safe.

We will go back to this in the next section, which was lectured later than it appeared in these notes, with proofs not fully rigorous in lectures.

3 Integral theorems

3.1 Divergence theorem

3.1.1 Statement and proof

Theorem. (3D divergence theorem) Let R be the closure of a bounded open set in $\mathbb{R}^3$ and suppose ∂R is a finite union of 2-manifolds glued together at the edges, and suppose

$$F(x, y, z) = (a(x, y, z), b(x, y, z), c(x, y, z))$$

has continuous and bounded first partial derivatives in R, then

$$\iiint_R div(F)\, dxdydz = \iint_{\partial \mathrm{R}} F.n\; dA$$

where n is normal to the boundary of R pointing outwards.

We need the additional condition that if ∂R is defined piecewise then the boundary between any 2 surfaces must be continuously differentiable, or at least have its δ-neighbourhood be $o(\delta)$. We will see what this means in the proof.

Proof. Since these integrals are of bounded continuous functions over bounded regions we can do things like taking limits as we refine partitions.

By a partition of unity (See analysis lemmas) we just need to prove that for each Q in R, there is an ε_Q such that for all such f supported only in $B_Q(\varepsilon_Q)$ the formula holds. Small note: We define our starting function as the bump function supported on [-1, 1] composed with the absolute value so it is supported on a ball instead of the rectangle.

First, suppose Q is interior to R, then we can pick a cube I contained in the interior of R centered at Q and suppose the support of f is contained within that cube. If we do this, then clearly

$$\iiint_R div(F)\, dxdydz = \iiint_I div(F)\, dxdydz$$

since $div(F)$ is 0 outside the cube so adding that to the integral changes nothing. Also, since the support of F is outside the boundary, $\iint_{\partial \mathrm{R}} F.n\; dA = 0$ in this case so we just need to show that $\iiint_I div(F)\, dxdydz = 0$, which is true by the fact that we can swap integrals around as I'll explain in a moment (we have continuity on a bounded interval). Therefore we have

$$\iiint_I a_x + b_y + c_z dxdydz = \iiint_I a_x dxdydz + \iiint_I b_y dydxdz + \iiint_I c_z dzdxdy$$

Therefore the fundamental theorem of calculus implies

$$\iiint_I a_x + b_y + c_z dxdydz = \iint 0dydz + \iint 0dxdz + \iint 0dxdy$$

since a,b,c is 0 on either side where the integrals are each inside a square that is a cross section of I, and basically we get 0 as desired.

Now suppose Q is on the boundary of R and not at the edge of one of the smooth pieces of the boundary. Choose suitable ε_Q such that we have a regular continuously differentiable parametrization (which is actually the same as the kind of patch piece we need for the manifold definition – a regular parametrization has non-singular derivative)

$$X : U\,(u,v) \to B_Q\,(\varepsilon_Q) \cap \partial\text{R}$$

We can rotate the space if necessary such that the normal vector to ∂R at Q (ie $X_u \times X_v$) is (0,0,1) and X_u and X_v are in linearly independent directions and renaming u and v to linear combinations of themselves change the parametrization such that $X_u = (0,0,1)\,,\ X_v = (0,1,0)$ at some u_0, v_0,when projected onto the x-y plane.

$$X : U\,(u,v) \to B_Q\,(\varepsilon_Q) \cap \partial\text{R}$$

then by continuity of $X_u \times X_v$ has no non-zero z component in some neighbourhood of Q with a refined ε_Q (needed so that X_u and X_v are not vertical.

Write

$$X\,(u,v) = (f_1\,(u,v)\,,\ f_2\,(u,v)\,,\ f_3\,(u,v))$$

then by how we rotated the space, we know that the function $Y\,(u,v) = (f_1\,(u,v)\,,\ f_2\,(u,v))$ has a local continuously differentiable inverse for some refined ε_Q by the inverse function theorem (since its derivative is the identity which is non-singular at (u_0, v_0) so we can write $g \circ X\,(u,v) = (u, v,\ g \circ f_3\,(u,v))$ for g the inverse of Y. So we can safely say that X sends u and v to $(u, v, \phi\,(u,v))$ as there is such an X.

Near the bottom of our 3D region, it will be locally the case that if we take some (x,y,z) it is in the inside of R if $z > \phi\,(u,v)$, and near the top the opposite holds. The point is, one of these always holds for geometrical reasons. Without loss of generality we will assume the "near the top" case. A standard computation gives

$$n = \frac{X_u \times X_v}{|X_u \times X_v|} = \left(\frac{-\phi_u}{\sqrt{1+\phi_u^2+\phi_v^2}}, \frac{-\phi_v}{\sqrt{1+\phi_u^2+\phi_v^2}}, \frac{1}{\sqrt{1+\phi_u^2+\phi_v^2}} \right)$$

In the analyisis lemmas document we discussed integrals over manifolds and why they are well defined. By picking $\alpha = (u, v, \phi\,(u,v))$ we have

$$V\,(D\alpha) = \sqrt{Det\left((D\alpha)^T\,(D\alpha)\right)} = \sqrt{Det\left(\begin{pmatrix} 1 & 0 & \phi_u \\ 0 & 1 & \phi_v \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \phi_u & \phi_v \end{pmatrix}\right)}$$

$$= \sqrt{Det\begin{pmatrix} 1+\phi_u^2 & \phi_u\phi_v \\ \phi_u\phi_v & 1+\phi_v^2 \end{pmatrix}} = \sqrt{Det\,(1+\phi_u^2+\phi_v^2)}$$

Therefore by the formula for surface integrals over manifolds,

$$\iint_{\partial R} \frac{c}{\sqrt{1+\phi_u^2+\phi_v^2}} dA = \iint_U c\,(u, v, \phi\,(u,v))\,dudv$$

We can now (by stretching or rotating our heads) redo our parametrization to have our u and v derivative vectors perpendicular and suppose our rotation made them the x and y axes respectively, then we have

$$\iint_{\partial R} \frac{c}{\sqrt{1+\phi_u^2+\phi_v^2}} dA = \iint_U c\,(x, y, \phi\,(x,y))\,dxdy$$

But also, by the fundamental theorem of calculus, and the fact that U is assumed to contain all points such that these things do not vanish (so in particular they vanish at minus infinity),

$$\iint_{\partial R} \frac{c}{\sqrt{1+\phi_u^2+\phi_v^2}} dA = \iint_U c\,(x, y, \phi\,(x,y))\,dxdy = \iint_{\mathbb{R}^2} \int_{-\infty}^{\phi(x,y)} c_z dzdxdy = \iiint_R c_z dxdydz$$

Therefore since the left integral is the third component of F.n and the right is the third component of div(F).

This proves 3D divergence theorem for this case by adding each component and doing a similar argument for each component.

We now need a lemma which allows us to extend to the piecewise smooth case. You would think well the whole corner region is measure 0 so it's fine, but the reason it breaks is subtle. It is ok for the sureface bit, but as our partition of unity gets finer, its derivative goes off to infinity, and we are doing stuff with its derivative. Therefore you cannot conclude anything. We will need to do this carefully.

The idea is to take the boundary of the boundary of your 3D region and take the set of points within δ of it. The strategy of the proof is as follows:

1. Take the δ-neighbourhood of the boundary of the boundary (we will use the fact that this is a compact set and also a union of 1-manifolds)

2. Construct a smooth function which is 0 on our boundary of the boundary and 1 outside the neighbourhood whose derivative is bounded by $\frac{C}{\delta}$ for some constant C. Take F and 1-F to be another partition of unity, conclude from previously that for our original function times 1-F the divergence theorem actually holds, and show that the F part $\to 0$ as $\delta \to 0$

3. Demonstrate that the volume of the δ neighbourhood as we shrink δ is $O\left(\delta^2\right)$, so we conclude that the integral goes to 0.

In step 1 there is not really anything to prove, so

Proof of step 2:

Let η be a function which integrates to 1 and is smooth and supported on the open ball with radius 1, for example the function

$$\begin{cases} e^{-\frac{1}{1-|u|^2}} \; if \;\; |u|<1 \\ 0 \text{ otherwise} \end{cases}$$

divided by its total integral.

Also let d(x) be the nearest distance to our bad set K. Then by a triangle inequality argument, we have $d\left(x\right) \leq |x-y|$ for y in our bad set which means

$$d\left(x\right) \leq |x-x'| + |x'-y| \leq |x-x'| + d(x')$$

so we conclude that $d\left(x\right) - d\left(x'\right) \leq |x-x'|$ and by a symmetric argument we can extend the result to get that in fact $|d\left(x\right) - d\left(x'\right)| \leq |x-x'|$

Define

$$F\left(x\right) = \int_{\mathbb{R}^3} d\left(z\right)\left(\frac{\delta}{4}\right)^{-n} \eta\left(\frac{4(x-z)}{\delta}\right) dz = \int_{\mathbb{R}^3} d\left(x-z\right)\left(\frac{\delta}{4}\right)^{-n} \eta\left(\frac{4(z)}{\delta}\right) dz$$

(the 2 integrals are equal as one is just the other but shifted and rotated). We want to show that F satisfies our requirements.

It is clearly smooth since η is smooth (we can differentiate under the integral sign since everything is continuous on a closed interval). Furthermore its derivative is bounded by 1 since changing x by a vector with size h makes the difference

$$\int_{\mathbb{R}^3} \left(d\left(x-z+hv\right) - d\left(x-z\right)\right)\left(\frac{\delta}{4}\right)^{-n} \eta\left(\frac{4(z)}{\delta}\right) dz$$

where v is a unit vector, so this is bounded by

$$\int_{\mathbb{R}^3} |h|\left(\frac{\delta}{4}\right)^{-n} \eta\left(\frac{4(z)}{\delta}\right) dz = |h|$$

by previous discussion.

Now observe that there exists a smooth function $\phi : \mathbb{R} \to [0,1]$independent of δ, such that:

1. $\phi(s) = 0$for $s \le 0$,

2. $\phi(s) = 1$for $s \ge 1$,

3. $|\phi^{'}(s)| \le M$ for some constant $M > 0$ and all s.

This is well known by analysis lemmas stuff, it's standard.

Now $\phi\left(\frac{\left(F(x)-\frac{\delta}{4}\right)}{\frac{\delta}{2}}\right)$ is a function that meets all of our requirements.

Proof of step 3:

We use the fact that our set is compact and is the image of [0,1] under some smooth function γ. All functions we use the theorem on will satisfy this.

Now by smoothness + the extreme value theorem, the derivative never exceeds some M in size. In particular,

$$\mid\gamma\left(s\right)-\gamma\left(t\right)\mid\le M|s-t|$$

Now partition [0,1] such that each step is of size at most $h \coloneqq \frac{\delta}{2M}$, and the number of intervals is $\le \frac{2M}{\delta}+1$. Now for each t there is always some t_i such that

$$\left|\gamma\left(t\right)-\gamma\left(t_i\right)\right|\le M\left|t-t_i\right|\le Mh=\frac{\delta}{2}$$

Therefore every point on our bad set lies within a distance $\frac{\delta}{2}$ of one of our sample points.

Now by the triangle inequality, this implies that if we take the union of balls of radius $\frac{3\delta}{2}$ around our sample points, it includes our δ neighbourhood but its volume is less than $\frac{M}{\delta}O\left(\delta^3\right)$ so we are done.

□

Note that the 2D divergence theorem has the same proof, but simpler. We note that we just take a ball around the corners so extending to the piecewise case is much easier.

We are now ready to return to the spherical polar coordinates world, but first we do this more generally.

Suppose we have r_u, r_v, r_w perpendicular and $h_u, h_v,\ h_w$ their lengths. Suppose we have a function vector field F and then write it as $F = F_u e_u + F_v e_v + F_w e_w$ where e is the normalized (ie unit) vector given by $e = \frac{r}{h}$.

3.1.2 Use to derive divergence formula

Here we assume the divergence to be continuous and the formulae for the basis vectors to be twice differentiable, meaning the main vector field is 3 times differentiable and then we let ΔV be the region defined by the cuboid

$$[u_0, u_0+du]\times[v_0, v_0+dv]\times[w_0, w_0+dw]$$

We just assume everything is infinitely differentiable in this course so that we can make the error terms involve more d terms than the main contribution and therefore can safely take limits. Here we assume we are looking at some cuboid-like region in coordinate space, not the tangent cuboid – we will put bounds on the error for sufficiently smooth (lets just say infinitely differentiable) functions. For all coordinate systems we will use it is clear that these regions are nice enough for the divergence theorem. Then the divergence is given (by the 3D divergence theorem) by

$$\lim_{du,dv,dw\to 0}\frac{1}{vol\left(\Delta V\right)}\iint_{\partial(\Delta V)}F.\ n\ dS$$

The volume of this box is

$$h_u h_v h_w du dv dw$$

evaluated at some point u', v', w' in the "box", by a mean value theorem, the ratio between this and the one evaluated at u, v, w goes to 1 by smoothness.

Now lets look at what goes through the boundary at the two faces perpendicular to e_u. The area of these faces are

$$h_v h_w dvdw + O\,(\text{terms with 3 d thingys multiplied together})$$

Since whatever you use to parametrize the rectangle-like region will be sufficiently smooth by assumption.

The outward normals are $-e_u, +e_u$. We are now integrating this over the two faces:

$$\left(F_u\left(u+du, v, w\right)h_v\left(u+du\right)h_w\left(u+du\right)\right)dvdw - \left(F_u\left(u,v,w\right)h_v\left(u\right)h_w\left(u\right)\right)dvdw$$

Now, by a mean value theorem, the integral of this is exactly the integral

$$\left(F_u\left(u+du, v^*, w^*\right)h_v\left(u+du\right)h_w\left(u+du\right)\right)dvdw - \left(F_u\left(u,v^*,w^*\right)h_v\left(u\right)h_w\left(u\right)\right)dvdw$$

for some v* and w* in the "rectangle". By the mean value theorem again, this time for the derivative, this is exactly du times the following partial derivative evaluated at some point u**:

$$\frac{\partial}{\partial u}F_u h_v h_w$$

Thus, our integral is

$$\frac{\partial}{\partial u}F_u h_v h_w$$

evaluated at (u^{**}, v^*, w^*), which approaches (u, v, w).

In the end, the total integral, by considering also from the other 4 faces, in the limit, is

$$\left[\frac{\partial}{\partial u}\left(h_v h_w F_u\right) + \frac{\partial}{\partial v}\left(h_u h_w F_u\right) + \frac{\partial}{\partial w}\left(h_u h_v F_u\right)\right]dudvdw$$

so dividing by the volume and taking the limit we get the following result:

$$div\left(F\right) = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(h_v h_w F_u\right) + \frac{\partial}{\partial v}\left(h_u h_w F_v\right) + \frac{\partial}{\partial w}\left(h_u h_v F_w\right)\right]$$

For sufficiently smooth functions. We are safe when they are infinitely differentiable, which they always are in this course except on measure 0 sets that don't affect stuff.

For the curl calculation, we need more theorems. After this, we will apply this to spherical polar coordinates as an example.

3.1.3 Green's theorem

Theorem. (Green's theorem) n Let R be a region with piecewise smooth boundary and let P and Q be functions from $\mathbb{R}^2 \to \mathbb{R}$, then $\oint_{\partial R} Pdx + Qdy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$. I proved this for rectangles in my Differential Equations notes (See level 7).

Proof. This follows directly from the 2D divergence theorem.

Define $F \coloneqq (Q, -P)$ to be another vector field, then the 2D divergence theorem implies that

$$\oint_{\partial R} F.n\ dr = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA$$

Therefore we just need to show $\oint_{\partial R} F.n\ dr = \oint_{\partial R} Pdx + Qdy$ with the path integral in the counterclockwise direction.

If $x(t)$ and $y(t)$ is a parametrization of our curve at some point, then the unit tangent is obtained by

$$\left(\frac{x'(t)}{\sqrt{x'(t)^2+y'(t)^2}},\frac{y'(t)}{\sqrt{x'(t)^2+y'(t)^2}}\right)$$

The outward normal is obtained by rotating this 90 degrees clockwise, ie it is

$$\left(\frac{y'(t)}{\sqrt{x'(t)^2+y'(t)^2}},\frac{-x'(t)}{\sqrt{x'(t)^2+y'(t)^2}}\right)$$

Therefore

$$\int F.ndr=\int\left[P\left(x\left(t\right),y\left(t\right)\right)\left[x'\left(t\right)\right]+Q\left(x\left(t\right),y\left(t\right)\right)\left[y'\left(t\right)\right]\right]dt$$

since the minuses cancel out on the P side. This is just

$$\int P\left(x\left(t\right),y\left(t\right)\right)dx+Q\left(x\left(t\right),y\left(t\right)\right)dy$$

which completes the proof.

□

Definition: We say a surface is orientbale if it has two sides. As an example, a mobius strip is not orientable because we can trace it and get to the other side.

Clearly, any closed loop or closed surface (meaning it encloses a region and does not have a boundary, not closed in the sense we are used to, or in the precise sense its patch is always open in the whole plane and not the half plane) has an interior and an exterior and we have an orientation that way as long as it is not self-intersecting. And yes, I know that the jordan curve theorem is hard to prove, but we don't need to prove it.

3.2 Stokes theorem

3.2.1 Statement and proof

Theorem. (Stokes theorem)

$$\oint_{\partial S}F.dr=\iint_S\left(\nabla\times F\right).n\ dA$$

on a 2 times continuously differentiable region S with piecewise continuously differentiable boundary and F having a continuous bounded derivative in this region. Here n is the unit normal in the standard orientation (ie, if the boundary of S is traced anticlockwise if you look at the surface from the direction that the outward normal is pointing).

To extend to S piecewise smooth we can do a sort of add up a bunch of integrals and the boundaries will be in opposite directions so we cancel trick.

Proof. Again, we only need to prove this on arbitrarily small patches by a partition of unity.

Again, we can do a rigid motion and parametrize the patch by $r\left(u,v\right)=\left(u,v,\phi\left(u,v\right)\right)$ with ϕ 2 times continuously differentiable by the inverse function theorem.

Note that

$$r_u=\left(1,0,\phi_u\right),\ r_v=\left(0,1,\phi_v\right),\ r_u\times r_v=\left(-\phi_u,-\phi_v,1\right)$$

Again, write

$$\iint_S\left(\nabla\times F\right).n\ dA=\iint_{r^{-1}(S)}\left(\nabla\times F(u,v,\phi\left(u,v\right)\right).\left(-\phi_u,-\phi_v,1\right)dudv$$

(n is rescaled by $Vol\left(D\phi\right)$)

Also, $dr = r_x dx + r_y dy$ by the chain rule as r is a function of x and y.

Therefore, (as usual considering only the part of

$$\oint_{\partial S} F.dr = \oint_{r^{-1}(\partial S)} F\left(r\left(x,y\right)\right).r_x dx + F\left(r\left(x,y\right)\right).r_y dy$$

If $F = (a, b, c)$ then

$$\oint_{\partial S} F.dr = \oint_{r^{-1}(\partial S)} P dx + Q dy$$

Where

$$P\left(x,y\right) \coloneqq a\left(x,y,\phi\left(x,y\right)\right) + c\left(x,y,\phi\left(x,y\right)\right)\phi_x(x,y)$$
$$Q\left(x,y\right) \coloneqq b\left(x,y,\phi\left(x,y\right)\right) + c\left(x,y,\phi\left(x,y\right)\right)\phi_y(x,y)$$

So by green's theorem,

$$\oint_{\partial S} F.dr = \iint_{r^{-1}(S)} Q_x - P_y dxdy$$

For functions supported in a sufficiently small neighbourhood.

Therefore the theorem is equivalent to

$$\iint_{r^{-1}(S)} Q_x - P_y dxdy = \iint_{r^{-1}(S)} \left(\nabla \times F(u,v,\phi\left(u,v\right)\right).\left(-\phi_u, -\phi_v, 1\right) dudv$$

$Q_x = b_x + b_z\phi_x + \left(c\phi_y\right)_x$ (Chain rule, in this algebra everything is evaluated at $\left(x, y, \phi\left(x,y\right)\right)$ where u is renamed to x and v is renamed to y)

$P_y = a_y + a_z\phi_y + \left(c\phi_x\right)_y$ (Chain rule)

$Q_x - P_y = b_x + b_z\phi_x + \left(c\phi_y\right)_x - a_y - a_z\phi_y - \left(c\phi_x\right)_y$ (No comment)

$Q_x - P_y = b_x + b_z\phi_x + c_x\phi_y + c\phi_{yx} - a_y - a_z\phi_y - c_y\phi_x - c\phi_{xy}$ (Product rule)

$Q_x - P_y = b_x + b_z\phi_x + c_x\phi_y - a_y - a_z\phi_y - c_y\phi_x$ (Symmetry of mixed partials applies because ϕ is 2x continuously differentiable)

So the theorem is equivalent to

$$\iint_{r^{-1}(S)} b_x + b_z\phi_x + c_x\phi_y - a_y - a_z\phi_y - c_y\phi_x\left(x,y,\phi\left(x,y\right)\right) dxdy = \iint_{r^{-1}(S)} \left(\nabla \times F(x,y,\phi\left(x,y\right)\right).\left(-\phi_x, -\phi_y, 1\right) dxdy$$

So it suffices to show

$$b_x + b_z\phi_x + c_x\phi_y - a_y - a_z\phi_y - c_y\phi_x\left(x,y,\phi\left(x,y\right)\right) = \left(\nabla \times F(x,y,\phi\left(x,y\right)\right).\left(-\phi_x, -\phi_y, 1\right)$$

Lets carefully expand the cross product

$$\left(\left(c_y - b_z,\ a_z - c_x,\ b_x - a_y\right)\left(x,y,\phi\left(x,y\right)\right).\left(-\phi_x, -\phi_y, 1\right)$$
$$= -\left(\phi_x\left(c_y - b_z\right)\left(x,y,\phi\left(x,y\right)\right) - \left(\phi_y\left(a_z - c_x\right)\left(x,y,\phi\left(x,y\right)\right) + \left(b_x - a_y\right)\left(x,y,\phi\left(x,y\right)\right)\right)$$

This is exactly what we want, so done, as a measure zero argument and/or cancelling edges does the piecewise smooth case for us.

□

3.2.2 Use to derive curl formula

We will do a similar strategy to find $curl\left(F\left(u_0,v_0,w_0\right)\right)$ as what we did for div.

Here is the idea, like for the divergence theorem the proper justification is from mean value tricks.

Use the following local form of stokes theorem:

$$\left(\nabla\times F\right).n=\lim_{dv,dw\to 0}\frac{1}{area(S)}\oint_{\partial S}F.dr$$

Where S is the little curvy rectangle $[v,v+dv]\times[w,w+dw]$ and $e_u=n$.

Again the area is h_vh_wdvdw and all errors are O of a term with an extra d in this proof, I won't bother writing it out.

The integral around the loop in the correct orientation:

Edge 1: Fix w, add v:

$$F.dr\approx F_ve_v.h_vdve_V=h_vF_v(v_0,w_0)dv$$

with small error terms.

We do this for other edges and get

$$(h_vF_v)(v,w)dv+(h_wF_w)(v+dv,w)dw-(h_vF_v)(v,w+dw)dv-(h_wF_w)(v,w)dw$$

Now, with error in terms involving 2 d terms, we get

$$(h_wF_w)(v+dv,w)\approx(h_wF_w)(v,w)+\frac{\partial}{\partial v}(h_wF_w)dv$$

$$\left(h_vF_v\right)\left(v,w+dw\right)\approx\left(h_vF_v\right)\left(v,w\right)+\frac{\partial}{\partial w}\left(h_vF_v\right)dw$$

We then get the integral as

$$\oint F\cdot dr\approx\left(\frac{\partial}{\partial v}\left(h_wF_w\right)-\frac{\partial}{\partial w}\left(h_vF_v\right)\right)dvdw$$

We again divide by the area and take limits to get

$$\left(\nabla\times F\right).e_u=\frac{1}{h_vh_w}\left(\frac{\partial}{\partial v}\left(h_wF_w\right)-\frac{\partial}{\partial w}\left(h_vF_v\right)\right)$$

Packaging these into a determinant we get the following result:

$$\nabla\times F=\frac{1}{h_uh_vh_w}\begin{vmatrix}h_ue_u & h_ve_v & h_we_w\\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w}\\ h_uF_u & h_vF_v & h_wF_w\end{vmatrix}$$

Assuming our set is right handed.

This is a **NOTATIONAL TRICK** and you MUST expand by the first row and multiply means to apply the differential operator. I am **NOT** claiming that the differential operator is something you can multiply by like those "physicists". The absolute value here represents the DETERMINANT.

4 Back to coordinate systems

Now suppose f is a scalar field again. We want to find a formula for $\nabla^2f=\nabla\cdot\left(\nabla f\right)=div\left(grad\left(f\right)\right)$

The idea is to apply the divergence formula to grad(f). We first need to find grad(f).

Now the directional derivative D_af is $\nabla f.a$.

Along the coordinate curve,

$$r\left(u+du,v,w\right)=r\left(u,v,w\right)+duh_ue_u+o(du)$$

Therefore the directional derivative of a smooth (smooth meaning infinitely differentiable) function in the unit direction e_u is $\frac{1}{h_u}\frac{\partial}{\partial u}f$ evaluated at (u, v, w).

We therefore get

$$\nabla f = \left(\frac{1}{h_u}\frac{\partial f}{\partial u}\right) e_u + \left(\frac{1}{h_v}\frac{\partial f}{\partial v}\right) e_v + \left(\frac{1}{h_w}\frac{\partial f}{\partial w}\right) e_w$$

Now apply the divergence formula

$$div\,(F) = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(h_v h_w F_u\right) + \frac{\partial}{\partial v}\left(h_u h_w F_u\right) + \frac{\partial}{\partial w}\left(h_u h_v F_u\right)\right]$$

To grad(f). We end up with this result:

$$\nabla^2 f = \frac{1}{h_u h_v h_w}\left\{\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right\}$$

And so we are done.

Now lets do an example of how to apply this to spherical coordinates.

Suppose F is the identity function on vectors so F(v)=v and we want to find its divergence while working in spherical coordinates. In this case,

$$F = re_r, h_r = 1, h_\theta = \text{r}, h_\phi = r\sin(\theta)$$

We therefore get (keeping in mind that $F_\theta, F_\phi = 0,\ F_r = r$),

$$div\,(F) = \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial r}\left(r^3\sin(\theta)\right) = \frac{3r^2\sin(\theta)}{r^2\sin(\theta)} = 3$$

Exactly as we expect.

[Lecture 6 ends]

5 Integrals

5.1 Scalar line integrals

The integral of something over a parametrized curve C with respect to t is as follows:

$$\int f\ ds = \int f\ V\,(dC)\ dt = \int f\left|\frac{dC}{dt}\right| dt = \int f\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

We say the natural parametrization is one where $\left|\frac{dC}{dt}\right|$ is always 1.

Example. The parametrization $(\cos(t), \sin(t), t)$ gives a helix (Figure 2).

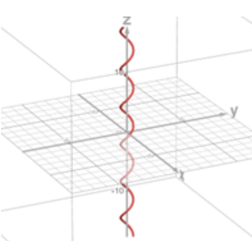


Figure 2

Example. See figure 3

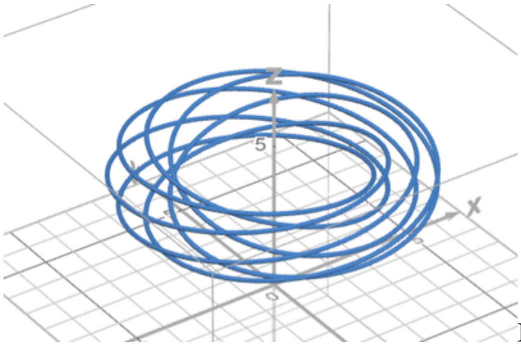


Figure 3

This knot is parametrized as follows:

$$((4+5\cos(t))\cos(7t), (4+5\cos(t))\sin(7t), 4+5\cos(t))$$

Now we will assume a curve is parametrized by arc length.

5.2 Curvature and the Frenet Serret equations

We will define the principal normal vector n as the normalized rate of change (whenever it is not 0) of the unit tangent vector t with respect to arc length. Ie, $\frac{d^2C}{ds^2}$ where s is an arc length parametrization. N is perpendicular to t because if t varies continuously then geometrically if its tail is fixed its tip is moving perpendicularly to t.

Proposition. (t, n, t×n) is clearly right handed and orthonormal and the parametrization is infinitely differentiable at the point. Then we have

$$\frac{dt}{ds} = kn, \frac{dn}{ds} = T\,(\mathrm{t}\times\mathrm{n}) - \mathrm{kt}, \frac{\mathrm{d}\,(\mathrm{t}\times\mathrm{n})}{ds} = -Tn$$

From now on we will write b to mean t×n

Proof. Let $k = \left|\frac{dt}{ds}\right|$, then the first equation is immediate. Here something ' means derivative wrt s.

Lets differentiate both sides of the equation t.b=0 so we get $0 = t'.b + t.b' = kn.b + t.b' = t.b'$

Therefore t is perpendicular to b' and b is perpendicular to b' since they vary continuously (again we are working with infinitely differentiable stuff in this course), which means b' is parallel to n. We therefore say $\frac{db}{ds} = -Tn$.

We now say

$$n = b \times t$$

$$\frac{dn}{ds} = b' \times t + b \times t' = -T\,(n \times t) + k\,(b \times n) = Tb - kt$$

So that completes the proof.

□

These are called the Frenet-Serret equations.

If k=0 then t'=0 and everything else is undefined.

We call k the curvature and T the torsion. K is the curvature because the more the curve is changing in the normal direction the more curved it will be.

Definition. The osculating plane is the plane through t and n. This therefore includes the tangent vector and its rate of change so it is a pretty good plane approximation to the curve, in some sense the best. Define the radius of curvature as $\frac{1}{k}$. This is because in a circle of radius r parametrized by length the second derivative of the motion does indeed (as one can check by parametrizing it as follows: $\left(rcos\left(\frac{t}{r}\right), rsin\left(\frac{t}{r}\right)\right)$ which is indeed by arc length and doing the derivatives, I will not go through the details) point inwards towards the circle with length $\frac{1}{r}$, so it is essentially the radius of the tangent circle.

If x is not parametrized by arc length we could roughly speaking divide through by $\left|\frac{dC}{dt}\right|$ to make it become parametrized by arc length, as we did above with the circle example above..

Example. Consider the curve $\left(\frac{\sqrt{2}}{2}t^2, t, t\right)$. Here is what that looks like (Figure 4):

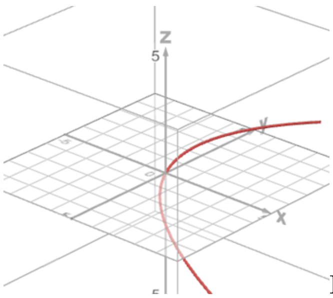


Figure 4

We see that at any point there are infinitely many planes tangent to the curve but there is in some sense a best one, in this case the one that is the plane the entire curve lies on, but in other cases it would be more local and the entire curve would not lie on one plane like this. Here n points into the parabola and b points outwards. We will now try to calculate the curvature and the torsion.

If $x\left(t\right) = \left(\frac{\sqrt{2}}{2}t^2, t, t\right)$ then $x' = \left(\sqrt{2}t, 1, 1\right)$ and if s is the arc length then $\frac{ds}{dt} = |x'| = \sqrt{2(1+t^2)}$. Now we have that the normalized tangent vector is $\frac{1}{\sqrt{2(1+t^2)}}\left(\sqrt{2}t, 1, 1\right)$. The derivative of this is not something I will go through the painful calculation of, but the answer turns out to be

$$\frac{1}{\sqrt{2}(1+t^2)^{\frac{3}{2}}}\left(\sqrt{2}, -t, -t\right)$$

Now divide this by $\frac{ds}{dt} = \sqrt{2(1+t^2)}$ so we have the derivative of the tangent with respect to s instead of t, which gives us

$$\frac{1}{2(1+t^2)^2}\left(\sqrt{2}, -t, -t\right)$$

The length of this is

$$\frac{1}{\sqrt{2}(1+t^2)^{\frac{3}{2}}}$$

So that is the curvature.

Therefore the radius of the tangent circle grows roughly like t^3.

Also can I mention how much I HATE whoever decided to use t for both the tangent and the parametrization variable they genuinely made me waste so much time trying to understand an already difficult concept because I kept getting confused.

[Lecture 7 ends]

5.3 Scalar surface integrals

A surface is NOT any map from $\mathbb{R}^2 \to \mathbb{R}^3$ even though the lecturer claimed so – if the lecturer is right then any non-empty subset of $\mathbb{R}^3$ would be a surface (This is a good numbers and sets exercise to show this!). It is such a map that we can differentiate or a union of such maps.

Example. A hemisphere can be parametrized by

$$\begin{pmatrix} \sin(u)\cos(v) \\ \sin(u)\sin(v) \\ \cos(u) \end{pmatrix}$$

Where u goes from $\left(\frac{\pi}{2}, \pi\right]$ and v goes from $[0, 2\pi)$.

Recall that the integral of a surface parametrized by α is given by $\int_A (f \circ \alpha) V(D\alpha)$ from the analysis lemmas document. Therefore to find the volume we just integrate

$$\int_A V(D\alpha) = \int_A \left| \frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v} \right|$$

Since the integral of 1 is the same as the integral of $1 \circ \alpha$. Similarly we know that volume is given by the triple product. If we have a volume parametrized by u, v and w then we integrate dV which is $\left| \frac{\partial \alpha}{\partial u} . \left(\frac{\partial \alpha}{\partial v} \times \frac{\partial \alpha}{\partial w} \right) \right| dudvdw$. We saw above that hemispheres are indeed parametrized from a rectangle, and we can safely integrate it since it is indeed continuously differentiable everywhere but the origin where the absolute value (r) is not, but this is a measure 0 set.

5.4 Vector line integrals

We define the integral of a vector field along a smooth curve C by

$$\int F.dx = \int_a^b F(x(t)) . \frac{dx}{dt} dt$$

Where the curve is parametrized by the function x from x(a) to x(b).

Another type of curve integral is similar, it is given by

$$\int F \times dx = \int_a^b F(x(t)) \times \frac{dx}{dt} dt$$

The integral of a scalar along a curve is given by

$$\int f(x) \, ds = \int_a^b f(x(t)) \left| \frac{dx}{dt} \right| dt$$

If C is a closed curve then the line integral around the whole curve is denotes by $\oint_C F.dx$

Example. We often denote this like

$$\int (x \text{ component of } F) \, dx + (y \text{ component of } F) \, dy + (z \text{ component of } F) \, dz$$

Consider the vector field $F = (y, -x, 0)$ on the straight line joining $(-1, 0, 0)$ to $(1, 0, 0)$. Then parametrize the curve as x=(t,0,0) for t is [-1, 1]. Then $F(x(t)) = (0, -t, 0)$ and $x' = (1, 0, 0)$ so the integral is $\int_{-1}^1 (0, 1, 0) . (1, 0, 0) \, dt = 0$.

Example. The integral of $(3x^2, 2y, 0)$ along the same line: We get $F(x(t)) .x' = \left(3t^2, 0, 0\right) .(1, 0, 0)$ and so the integral is $\int_{-1}^1 3t^2 dt = 2$.

Example. Suppose we now have a curve parametrized by $\left(t - 1,\ t(t-2),\ t\left(t^2 - 4\right)\right)$ for t ranging from 0 to 2, and we will evaluate the integral of $F = (y, -x, 0)$ along this curve. This is as follows:

$x' = (1,\ 2t - 2,\ 3t^2 - 4)$ and $F(x(t)) = (t(t-2), 1-t, 0)$. We could calculate the dot product and integrate it but we're lazy.

Example: Now lets integrate $(3x^2, 2y, 0)$ along the new curve $\left(t - 1,\ t(t-2),\ t\left(t^2 - 4\right)\right)$. We have

$F(x) = \left(3(t-1)^2, 2t(t-2), 0\right)$ and x' as in the previous example. If we dot them we get the cubic polynomial $4t^3 - 9t^2 + 2t + 3$ which we can integrate from 0 to 2.

Example. Consider the function $\left(y^2 e^x, e^y + 2ye^x, 0\right)$ along the curve $x(\theta) = (\text{acos}(\theta), bsin(\theta), 0)$ for ϑ from 0 to $\frac{\pi}{2}$. We get

$$F(x(\theta)) = \left(b^2 \sin^2(\theta) e^{\text{acos}(\theta)}, e^{b\sin(\theta)} + 2b\sin(\theta) e^{\text{acos}(\theta)}, 0\right)$$

And

$$x'(\theta) = (-\text{asin}(\theta), b\cos(\theta), 0)$$

And we could take the dot product and integrate.

We end up with

$$\int F.dx = \int_0^{\frac{\pi}{2}} -ab^2\sin^3\theta e^{a\cos\theta} + b\cos\theta e^{b\sin\theta} + 2b^2\sin\theta\cos\theta e^{a\cos\theta} d\theta$$

The second term is just the derivative of $e^{b\sin\theta}$ and the rest can be resolved with the substitution $u = \cos\theta$. You eventually get $e^b - 1 + b^2$.

Around the whole ellipse we will be going from 0 to 2π and the whole integral will go to 0.

Proposition. If we reparametrize the same curve in a continuously differentiable bijective manner such that the derivative is never singular, then the integral does not change.

Proof. Take two bijective C^1 (which means continuously differentiable) regular parametrisations of the same oriented curve C:

$$\gamma : [a, b] \to \mathbb{R}^n, \tilde{\gamma} : [\alpha, \beta] \to \mathbb{R}^n$$

with $\gamma'(t) \neq 0$ and $\tilde{\gamma}'(s) \neq 0$ everywhere, and both are bijections onto C. Therefore they have C^1 inverses, and the inverses are global and not local since they are bijections (The idea is we're dealing with compact sets so we can patch finitely many local inverses together).

Key structural fact: There exists a C^1 bijection (a reparametrisation)

$$\phi : [\alpha, \beta] \to [a, b]$$

such that

$$\tilde{\gamma}(s) = \gamma(\phi(s))$$

Now complete the line integral using $\tilde{\gamma}$:

$$\int_C F.dx = \int_\alpha^\beta F(\tilde{\gamma}(s))\tilde{\gamma}'(s)\,ds$$

Differentiate $\tilde{\gamma}(s) = \gamma(\phi(s))$ using the chain rule:

$$\tilde{\gamma}(s) = \tilde{\gamma}(\phi(s))\phi'(s)$$

Substitute that into the integral:

$$\int_C F.dx = \int_\alpha^\beta F(\tilde{\gamma}(s))\tilde{\gamma}(\phi(s))\phi'(s)\,ds$$

Now do the 1D substitution $t = \phi(s)$. Since ϕ is a C^1 bijection, it's monotone, so subtitution is legitimate. Also, we can replace $\phi'(s)\,ds$ with dt (this is just integration by substitution). Then if ϕ is increasing, t runs from a to b, and

$$\int_\alpha^\beta F(\gamma(\phi(s)))\gamma'(\phi(s))\phi'(s)\,ds = \int_\alpha^\beta F(\gamma(t))\gamma'(t)\,dt$$

That right hand side is exactly the integral computed with γ. So the value is independent of which bijective C^1 regular parametrisation you use, provided it preserves orientation.

One tiny orientation footnote: If ϕ is decreasing the same substitution gives the same equality but with a minus sign. i.e. you've parametrised the same geometric curve with the opposite direction, so the line integral flips sign, exactly as it should.

□

[Lecture 8 ends]

Consider $F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}$. We say this is exact if there is a scalar field f such that $\nabla(F_1, F_2, F_3)$ is equal to the gradient of for some scalar field f, ie is conservative. The difference between this and what we did in differential equations is that now there are 3 variables.

Note that by exactness and the fundamental theorem of calculus, if C is a path from a to b, then $\int_C F.dx = f(b) - f(a)$ for any piecewise smooth path C.

Note that by stokes theorem, if F is smooth and $\nabla \times F = 0$ everywhere in a simply connected domain the line integral along any loop is 0 since we just take the line integral and make it the boundary of some manifold where the curl is 0 everywhere on that manifold (by simply connectedness). HOWEVER, there is a problem (we need to know we can actually make a smooth manifold) which I will address later in these notes. The converse is true because curl(grad(f)) is 0 for smooth fields by symmetry of mixed partials. i.e.

$$\nabla \times (\nabla f) = \begin{pmatrix} \partial_y f_z - \partial_z f_y \\ \partial_z f_x - \partial_x f_z \\ \partial_x f_y - \partial_y f_x \end{pmatrix} = \begin{pmatrix} f_{zy} - f_{yz} \\ f_{xz} - f_{zx} \\ f_{yx} - f_{xy} \end{pmatrix}$$

In the examples from last lecture we see that when we connected the same points on a different path we got different answers if there was non-zero curl and the same answers otherwise. We can also show that curl is 0 if we directly find a scalar field that our vector field is the gradient of.

In fact, $F = \left(y^2 e^x, e^y + 2ye^x, 0\right)$ from last lecture is the gradient of $e^y + y^2 e^x$ so we just need to find the change in that between the end points, which is far easier than doing the cos substitution or whatever. Doing $\left[e^y + y^2 e^x\right]_{(a,0,0)}^{(0,b,0)}$ indeed gives the same answer.

5.5 Area integrals

We know how to do an integral over a flat 2D region like

$$\int_A f(x,y)\, dA = \iint f(x,y)\, dxdy = \iint f(x,y)\, dydx$$

By level 6.2, this is ok if f is riemann integrable on a rectangle. (We stated it for continuous functions but the proof works for piecewise continuous functions).

[Lecture 9 ends]

Example: Suppose we want to integrate xy^2 over the triangle with vertices at (0,0), (1,1) and (1,-1) then we write

$$\int_0^1 \int_{-x}^{x} xy^2 dydx = \int_0^1 x \int_{-x}^{x} y^2 dydx = \int_0^1 x \left[\frac{1}{3}y^3\right]_x^{-x} dx = \int_0^1 \frac{2}{3}x^4 dx = \frac{2}{15}$$

Since for each fixed x, y goes from -x to x within the triangle.

Note that for riemann integrable functions the lebesgue integral agrees with the riemann integral so we can apply theorems about integral swaps or dominated convergence.

We could also calculate

$$\int_{-1}^{1} \int_{|y|}^{1} xy^2 dxdy = \int_{-1}^{1} y^2 \left[\frac{1}{2}x^2\right]_{|y|}^{1} dy = \int_{-1}^{1} \frac{1}{2}y^2\left(1 - y^2\right) dy = \frac{2}{15}$$

It's almost like math works.

If A is a rectangle $[a,b] \times [c,d]$ and we are integrating a function of the form g(x)h(y) the result is just the product $\int_a^b g(x)\, dx \int_c^d h(y)\, dy$.

5.6 Change of variables for area and volume integrals

Recall that if we have a continuously differentiable bijection with continuously differentiable inverse from the compact area A to the area A', and this bijection is from x, y coordinates before and u, v coordinates after and we are integrating a continuously differentiable function that can be extended to a continuously differentiable function in an open region around A, and furthermore we can write $x(u,v)$ and $y(u,v)$ to represent the inverse, then we have

$$\iint_A f(x,y)\, dxdy = \iint_{A'} f\left(x(u,v),\ y(u,v)\right) \left| Det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| dudv$$

Example. Lets integrate y^2 over the semicircle $\{y \geq 0 \cap x^2 + y^2 \leq R^2\}$. Clearly f can be extended to a continuously differentiable function in an open set around the semicircle. Lets change to polar coordinates.

Uhh ok we need to cut out a small semicircle containing the origin and take a limit because of the hypotheses of the version of the theorem that we know ok fine we can do that because our functions are bounded and stuff lets just do it and lets complain about how the lecturer should have done the same.

The absolute value of the determinant turns out to be $\left| Det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right| = r$

And so we need to find $\int_{[0,R]\times[0,\pi]} r^2 \sin^2\theta |r| dr d\theta$. This is separable so we work out

$$\int_0^R r^3 dr \int_0^\pi \sin^2\theta d\theta = \frac{1}{8}\pi R^4$$

We could get an intuition for the change of variables theorem by thinking about tiny area elements or whatever and it works but it is very non-rigorous.

By the chain rule, we get the intuitive result

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} & \frac{\partial v}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

There is a questionable notation convention where we would write, for example the determinant of the above matrix, as $\frac{\partial(x,y)}{\partial(\xi,\eta)}$.

[Lecture 10 ends]

Example. By the previous calculation we have a sort of chain rule for these $\frac{\partial(x,y)}{\partial(\xi,\eta)}$ things. In particular we have

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{\frac{\partial(\xi,\eta)}{\partial(x,y)}}$$

Let $u = x^2 + y^2, v = x^2 - y^2$. Now $x = \sqrt{\frac{1}{2}(u+v)},\ y = \sqrt{\frac{1}{2}(u-v)}$

$\frac{\partial(x,y)}{\partial(u,v)}$ turns out to be $-\frac{1}{8xy}$.

However, in this case it is clearly much easier to calculate $\frac{\partial(u,v)}{\partial(x,y)} = -8xy$ as no square roots are involved.

Now lets calculate $\int_A x^3 y dA$ where A is the area between the hyperbolae $x^2 - y^2 = 1,\ x^2 - y^2 = 2$, the line y=0 and the circle $x^2 + y^2 = 4$.

Ie, this area (See figure 5)

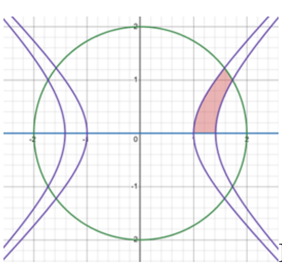


Figure 5

Note that here we have $1 < v < 2$ and the boundary for u is when $u = v$ (the x-axis) or $u = 4$ (the circle).

Now we get (since we have a continuously differentiable bijection with non-zero determinant everywhere and our function is extendable to a function on an open set outside our compact set)

$$\int_A x^3 y dA = \int_{A'} x^3 y \left| -\frac{1}{8xy} \right| dudv = \frac{1}{8}\int_{A'} x^2 dudv = \frac{1}{8}\int_{v=1}^{2}\int_{u=v}^{4} x^2 dudv$$

$$= \frac{1}{8}\int_{v=1}^{2}\int_{u=v}^{4}\frac{1}{2}\left(u+v\right)dudv = \frac{1}{8}\int_{v=1}^{2}\left[\frac{1}{4}u^2+\frac{1}{2}uv\right]_{v}^{4}dv = \frac{1}{8}\int_{v=1}^{2}4-\frac{1}{4}v^2+2v-\frac{1}{2}v^2dv$$

$$= \frac{1}{16}\int_{v=1}^{2}8+4v-\frac{3}{2}v^2dv = \frac{21}{32}$$

Volume integrals in 3D are basically the same thing.

Note that if we integrate wrt z first then y then x, then we need to work out the limits for the inner variables in terms of the outer variables.

As an example in 2D, say we want to integrate over the triangle with vertices (0,0), (0,1) and (1,0) in dx dy. Then if we say that y ranges from 0 to 1 and for each value of y, x goes from 0 to 1-y then it will be correct. We could try to do the converse by saying x goes from 0 to 1 and y ranges from 0 to 1-x, but we quickly run into trouble when once we solve the inner integral, x is no longer even defined.

We could do the integral in a different order but the point is the integration bounds can only depend on earlier variables.

We can do a volume integral of a vector component-wise, Ie, suppose we want to integrate $\left(z^2, 0, yz\right)$ over the cuboid $[-a,a]\times[-b,b]\times[-c,c]$.

Then we just integrate z^2 and yz over this cuboid separately. Since each can be written in the form f(x)g(y)h(z) and we are integrating over a cuboid we can split it into a product of 3 integrals and solve that, which we know how to do.

And we know how to do change of variables for volume integrals.

$$\int_V f\left(x,y,z\right)dxdydz = \int_V F\left(u,v,w\right)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dxdydz$$

As proven in the analysis lemmas document (not proven by splitting into parallelopipeds and taking limits by vibes, I don't like that method since it is not precise enough).

If we go to cylindrical polar coordinates, then we could compute the derivatives to get, recalling that

$$x = p*\cos\left(\theta\right),\ \ y = p*\sin\left(\theta\right)$$

the formula

$$\frac{\partial(x,y,z)}{\partial(p,\theta,z)} = \left|\det\begin{pmatrix}\frac{\partial x}{\partial p} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial p} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z}\end{pmatrix}\right| = \left|\det\begin{pmatrix}\cos\left(\theta\right) & -p*\sin\left(\theta\right) & 0\\ \sin\left(\theta\right) & p*\cos\left(\theta\right) & 0\\ 0 & 0 & 1\end{pmatrix}\right| = p$$

Here for example if we differentiate with respect to p then θ and z are assumed to be held constant.

In spherical polar coordinates, where $x = r*\sin\left(\theta\right)\cos\left(\phi\right),\ \ y = r*\sin\left(\theta\right)\sin\left(\phi\right), z = r*\cos(\theta)$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \left|\det\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{pmatrix}\right|$$

Actually there is a much nicer way to get the result than differentiating everything.

We want to find the volume of the parallelopiped generated by the three columns. But notice that this parallelopiped is exactly the cuboid whose sides are $e_i h_i$ – ie, the normalized rate of change of r is the length of the r column, and they are perpendicular, so it is just $|h_r h_\theta h_\phi|$ which is in fact $r^2\left|\sin\left(\theta\right)\right|$.

[Lecture 11 ends]

We can use this to find the volume of a sphere with radius less than R using a change of variables and noting that the origin is a single bad point that does not affect the result (We could remove a tiny sphere at the center and take a limit).

$$\int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2\left|\sin\left(\theta\right)\right|drd\theta d\phi = \int_0^R r^2dr*\int_0^\pi \sin\left(\theta\right)d\theta*\int_0^{2\pi}1d\phi = \frac{4}{3}\pi r^3$$

Example. Now we will consider a sphere of radius R with a cylinder of radius a removed from it. We can do this in cylindrical polar coordinates with a change of variables. Assume the cylidner is parallel to the z direction.

$$\int_{p=a}^{R}\int_{\theta=0}^{2\pi}\int_{z=-\sqrt{R^2-p^2}}^{\sqrt{R^2-p^2}} pdpd\theta dz = 2\pi * \int_{p=a}^{R}\int_{z=-\sqrt{R^2-p^2}}^{\sqrt{R^2-p^2}} pdzdp = 2\pi * \int_{p=a}^{R} 2p\sqrt{R^2-p^2}dp$$

The substitution $u = p^2$ lets us solve this integral and gives the answer

$$\frac{4}{3}\pi\left(R^2-a^2\right)^{\frac{3}{2}}$$

5.7 Surface flux integrals

We will again talk about surfaces. In this course we only consider surfaces that are sufficiently many times differentiable for all our theorems to hold, at least piecewise. We will say infinitely differentiable.

Now we will do a surface integral.

If we do surface areas on vectors we can integrate (as in the divergence theorem) $\int_S F.n\ ds$ which essentially measures how much projection of the vector field there is on the surface.

We sometimes write shorthand for this as $\int_S F.\ ds$.

Note that ds is just the area element – something which is a bit sloppy to say but we have made it rigorous at least for smooth functions and smooth parametrizations (in level 8.25).

We have $n = \pm\frac{x_u \times x_v}{|x_u \times x_v|},\ ds = |x_u \times x_v| dudv$

We get $\int_S F.n\ ds = \int_S F.\left(x_u \times x_v\right)\ dudv$

And we note that the sign depends on our choice of orientation for orientable surfaces.

And now the idea is we have turned surface integrals into area integrals which we know how to do.

Example. Lets find the surface area of the hemispherical shell where we parametrize by

$$\begin{pmatrix} R\sin(u)\cos(v) \\ R\sin(u)\sin(v) \\ R\cos(u) \end{pmatrix}$$

Where u goes from $\left(\frac{\pi}{2},\pi\right]$ and v goes from $[0,2\pi)$. R is just constant here.

Now $ds = x_u \times x_v = R^2\begin{pmatrix} \sin^2(u)\cos(v) \\ \sin^2(u)\sin(v) \\ \sin(u)\cos(v) \end{pmatrix}$. Its modulus is $R^2\sin(u)dudv$, we don't need to take absolute values because u does not range in such a way that its sine becomes negative.

So the surface area is just the integral over the rectangle $\left[\frac{\pi}{2},\pi\right] \times [0,2\pi]$ of $R^2\sin(u)$ which we can separate to get $R^2\int_{u=\frac{\pi}{2}}^{\pi}\sin(u)du\int_0^{2\pi}dv = 2\pi R^2$ as we know already.

[Lecture 12 ends]

Lets do the integral of $\int_S F.n\ ds$ where n points outwards, S is a sphere, and $F = \frac{a}{|x|^3}x$.

Now $ds = |x_u \times x_v|\, dudv = R^2 dudv$ because we have to do the volume thing.

Here n is $+\begin{pmatrix} \sin^2(u)\cos(v) \\ \sin^2(u)\sin(v) \\ \sin(u)\cos(v) \end{pmatrix}$

We do $+$ and not $-$ because we are choosing the outward normal.

Here S is $[0, \pi] \times [0, 2\pi]$ since it is the whole sphere

So F.n ds is

$$\frac{a}{R^3}\begin{pmatrix} R\sin(u)\cos(v) \\ R\sin(u)\sin(v) \\ R\cos(u) \end{pmatrix} . R^2 \begin{pmatrix} \sin^2(u)\cos(v) \\ \sin^2(u)\sin(v) \\ \sin(u)\cos(v) \end{pmatrix} dudv$$

Doing the dot product it simplifies a lot – all the R's go away and we do a lot of trigonometric identity magic and we end up with asin$(u)dudv$.

Now we integrate a$*$sin(u) as u goes from 0 to π and v goes from 0 to 2π.

We end up with $4\pi a$.

Now we will do another one of these vector flux integrals, this time through a plane. We will do the plane x-z=0 which we will parametrize by (u,v,u) and so

$$x_u \times x_v = (1, 0, 1) \times (0, 1, 0) = (-1, 0, 1)$$

We will pick the normal $(1, 0, -1)$.

Now we will integrate

$$\left(y^2 e^{-\left(y^2+z^2\right)}, e^{-y^4}, -x^2 e^{-\left(y^2+z^2\right)}\right)$$

So we will integrate over all of $\mathbb{R}^2$ the integral of

$$\left(y^2 e^{-\left(y^2+z^2\right)}, e^{-y^4}, -x^2 e^{-\left(y^2+z^2\right)}\right) .(1, 0, -1)$$

which is

$$\left(x^2 + y^2\right) e^{-(y^2+z^2)}$$

Converting to u,v coordinates we get $\int_{\mathbb{R}^2} \left(u^2 + v^2\right) e^{-(u^2+v^2)} dudv$. We just need to find this integral.

We will separate this and use some results from level 6 stats to get

$$\int_{\mathbb{R}^2} \left(u^2\right) e^{-(u^2+v^2)} dudv + \int_{\mathbb{R}^2} \left(v^2\right) e^{-(u^2+v^2)} dudv$$

$$\int_{\mathbb{R}^1} \left(u^2\right) e^{-(u^2)} du \int_{\mathbb{R}^1} e^{-(v^2)} dv + \int_{\mathbb{R}^1} e^{-(u^2)} du \int_{\mathbb{R}^1} v^2 e^{-(v^2)} dv$$

We use the fact that $\int_{\mathbb{R}^1} \left(u^2\right) e^{-(u^2)} du = \frac{\sqrt{\pi}}{2}$

Which we can get if we write it as $u * ue^{-u^2}$ and integrate it by parts by integrating ue^{-u^2} with the substitution $v = u^2$.

And we get that π is the result.

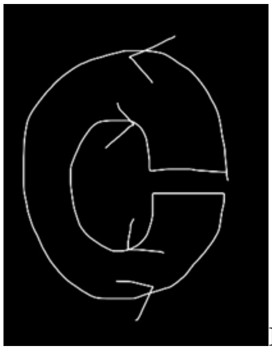
[Lecture 13 ends]

5.8 Use of integral theorems

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ with $P = -\frac{1}{2}y,\ Q = \frac{1}{2}x$. Now we will do an example of using Green's theorem. We have that $\int_A Q_x - P_y dxdy = Area\ of\ A = \pi ab$

By Green's theorem this is equal to $\int_{\partial A} Pdx + Qdy$. We can parametrize the boundary yadda yadda and the line integral gives πab.

We can use Green's theorem on areas with holes in them. We just have to traverse the hole clockwise and the exterior clockwise. The reason is because we can consider the image below and try to shrink the gap to 0 in the limit. See figure 6.

Figure 6

Or alternatively it is like just subtracting the inner area from the outer area since if we go clockwise we have to just subtract.

However the limit argument above does not require the domain to be simply connected.

Example of stokes theorem:

Consider a surface $z = \left(x^2 + y^2\right)^2$ with $0 \leq z \leq 1$ and we want to integrate the vector field $F = x^2 y$ over this surface. Then on cylindrical polar coordinates we get $z = p^4$ for ϑ going from 0 to 2π.

We will do the surface integral and the boundary integral to verify stokes theorem for this case and just generally practice calculating this stuff.

We have $\oint \left(x^2 y, 0, 0\right).n\ ds$.

In cylindrical polar coordinates we have $V = \begin{pmatrix} p * \cos(\theta) \\ p * \sin(\theta) \\ p^4 \end{pmatrix}$ where $0 \leq p \leq 1$ and $0 \leq \theta \leq 2\pi$.

Now we note a shortcut. We want to integrate a vector field F over a surface. In the past we would have said $\int F.n\ ds = \int F.\left(\frac{x_u \times x_v}{|x_u \times x_v|}\right)|x_u \times x_v|\, ds$ and we can actually cancel. In this case we want to integrate the curl of F, so we want

$$\oint (\nabla \times F).(V_p \times V_\theta)\, dpd\theta$$

Calculating the cross products and picking the sign so that the normal goes away from the origin, what you would think of as "outside" the bowl region even though it doesn't really have an inside or an outside (I won't go through all the details)

$$= \oint \begin{pmatrix} 0 \\ 0 \\ -x^2 \end{pmatrix}.\begin{pmatrix} 4p^4\cos(\theta) \\ 4p^4\sin(\theta) \\ -p \end{pmatrix} dpd\theta$$

$$= \iint \begin{pmatrix} 0 \\ 0 \\ -p^2\cos^2(\theta) \end{pmatrix}.\begin{pmatrix} 4p^4\cos(\theta) \\ 4p^4\sin(\theta) \\ -p \end{pmatrix} dpd\theta$$

$$= \iint p^3\cos^2(\theta)dpd\theta$$

If we integrate this where $0 \leq p \leq 1$ and $0 \leq \theta \leq 2\pi$ we get $\frac{\pi}{4}$.

On the other hand if we integrate y along the unit circle, we should get the same thing by Stoke's theorem. We get, if we suppose $v = \begin{pmatrix} \cos(t) \\ -\sin(t) \\ 1 \end{pmatrix}$, $F = \begin{pmatrix} x^2 y \\ 0 \\ 0 \end{pmatrix}$ (we go this way to be consistent with the direction of the normal for stokes theorem)

$$\int_{t=0}^{2\pi} F(v(t)).\frac{dv}{dt}dt = \int_{t=0}^{2\pi} \sin^2(t)\cos^2(t)dt = \frac{1}{4}\int_{t=0}^{2\pi} \sin^2(2t)dt = \frac{\pi}{4}$$

So we're done.

[Lecture 14 ends]

Example. We will verify stokes theorem for $F(x) = \left(y, z, x^2 + y^2\right)$ on the hyperbolic tube region given by the equation $z^2 + 1 = x^2 + 4y^2,\ 0 \leq z \leq \sqrt{3}$.

We will parametrize the surface by

$$v(z, \alpha) = \left(\sqrt{z^2+1}\cos(\alpha), \frac{1}{2}\sqrt{z^2+1}\sin(\alpha), z\right)$$

Here we have $0 \leq \alpha < 2\pi,\ 0 \leq z \leq \sqrt{3}$.

Lets evaluate the surface integral.

We want

$$\int_{\alpha=0}^{2\pi}\int_{z=0}^{\sqrt{3}} \pm F.(v_z \times v_a)\, dz d\alpha = \int_{\alpha=0}^{2\pi}\int_{z=0}^{\sqrt{3}} \pm F.(v_z \times v_a)\, dz d\alpha$$

We will pick the normal corresponding to the – sign. If you want you can check that I've done the changing variables and cross product correctly.

We note that

$$F = \begin{pmatrix} \frac{z}{\sqrt{z^2+1}}\cos(\alpha) \\ \frac{z}{2\sqrt{z^2+1}}\sin(\alpha) \\ 1 \end{pmatrix} \text{ and } v_z \times v_a = \begin{pmatrix} -\sqrt{z^2+1}\sin(\alpha) \\ \frac{1}{2}\sqrt{z^2+1}\cos(\alpha) \\ 0 \end{pmatrix}$$

Also,

$$\nabla \times F = \begin{pmatrix} 2y - 1 \\ -2x \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{z^2+1}\sin(\alpha) - 1 \\ -2\sqrt{z^2+1}\cos(\alpha) \\ -1 \end{pmatrix}$$

Now the surface integral is

$$\int_{\alpha=0}^{2\pi}\int_{z=0}^{\sqrt{3}} - \begin{pmatrix} \sqrt{z^2+1}\sin(\alpha) - 1 \\ -2\sqrt{z^2+1}\cos(\alpha) \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -\sqrt{z^2+1}\sin(\alpha) \\ \frac{1}{2}\sqrt{z^2+1}\cos(\alpha) \\ 0 \end{pmatrix} dz d\alpha$$

$$= \int_{\alpha=0}^{2\pi}\int_{z=0}^{\sqrt{3}} \left(-\frac{3}{2}\left(z^2+1\right)\sin(\alpha)\cos(\alpha) - \frac{1}{2}\sqrt{z^2+1}\cos(\alpha) + \frac{1}{2}z\right) dz d\alpha$$

This looks horrible but each of these terms are separable and luckily the integral of $\sin(\alpha)\cos(\alpha)$ and the integral of $\cos(\alpha)$ both go to 0 so we just end up with

$$2\pi \int_0^{\sqrt{3}} \frac{1}{2} z dz = \frac{3\pi}{2}$$

Note that the normal we chose has us going away from the origin. Now we have two boundary curves so we have to deal with it.

It is analagous to the case of Green's theorem where we have a hole. So we want to go in the standard direction on both. Let me break this down. You will need some visualization skills for this, so I will add a picture. See figure 7.

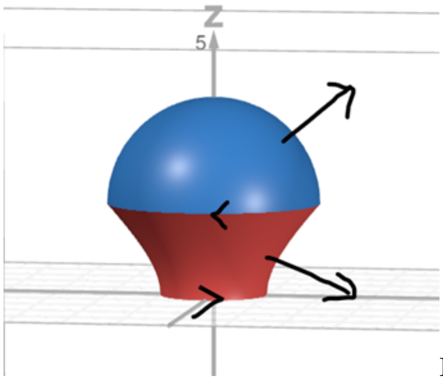


Figure 7

Red is what the part we are integrating over roughly looks like (not to scale). Blue is what I will call the extension for the purposes of this.

Suppose we extended the hyperbola over the $z = \sqrt{3}$ hole to fill it in. Then what we want is the integral over the extended hyperbola minus the integral of the extension. Therefore we want to traverse the boundary element at $z = 0$ in the usual way (the test for orientation says that if we walk around it with the surface on the left the normal should be upwards so we want to go in the direction of increasing α), and traverse the $z = \sqrt{3}$ boundary element in the reverse way that the hyperbola extension would want, which is actually the same way that the normal part of the hyperbola would want, which is the direction of increasing α.

Now we will find the two line integrals. For the $z = \sqrt{3}$ part we have the parametrization

$$v(t) = \begin{pmatrix} 2\cos(t) \\ \sin(t) \\ \sqrt{3} \end{pmatrix}$$

Now t goes from 0 to 2π and we find $\frac{dv}{dt}$ and we note that

$$F = \begin{pmatrix} \sin(t) \\ \sqrt{3} \\ 4\cos^2(t) + \sin^2(t) \end{pmatrix}$$

We can do the line integral and then do the same for the z=0 part and we indeed get $\frac{3\pi}{2}$. This verifies stokes theorem for this case.

Example of the divergence theorem:

Consider the vector field $\left(y, x, z^6\right)$ on the volume $|x| < R$, then the divergence inside the volume is given by $\int_{V_R} 6z^5 dV$ which is 0 by symmetry about $z = 0$ (since $6z^5 = -6(-z)^5$).

Now lets work out $\int_{\partial V_R} F.n \, dS$. Then we parametrize by sphericals with constant R. Therefore in the end we get $\int_{\theta,\phi \in [0,\pi]\times[0,2\pi]} (F) \, . \, (V_\theta \times V_\phi) \, d\theta d\phi$

$$= \int_{\theta,\phi \in [0,\pi]\times[0,2\pi]} \begin{pmatrix} R * \sin(\theta)\sin(\phi) \\ R * \sin(\theta)\cos(\phi) \\ R^6 * \cos^6(\theta) \end{pmatrix} .R^2 \sin(\theta) \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} d\theta d\phi$$

We end up with an area integral where all the terms vanish by a symmetry argument.

It did not matter which normal we take because we get 0.

This verifies the divergence theorem for this case.

Example of where the divergence theorem significantly simplifies a calculation: Lets find $\int_{\partial V} F.n \, dS$ where V is the half conical region given by

$$y^2 + z^2 = B^2(x+1)^2, 1 \le x \le 2$$

And F(v)=v.

The normal points outwards. But of course the surface integral would be horrible. But note that the divergence is 3 so the answer should just be 3 times the volume. See figure 8

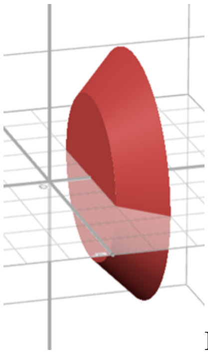


Figure 8

And we know from volume of cone formulas how to find 3x the volume of this.

If for whatever reason we want the surface integral over only the curved bit we can find the surface integral over the flat bits which is easier and then subtract the results.

[Lecture 15 ends]

The thing we did with greens theorem with a hole is the same thing we can do with the divergence theorem with a hole.

Proposition. If a vector field has 0 curl in a simply-connected open domain, then it is conservative.

Proof. We want to show that the integral over any piecewise smooth loop is 0 as that is equivalent to path independence which is equivalent to being conservative (we can define the a function as the integral from some fixed point on some piecewise smooth path, such paths always exist for the same reason as the argument we will do shortly, and take grad of that). However, by simply-connectedness, we can find a surface it encloses and apply Stokes theorem (0 curl means 0 integral along boundary). However, we are not done as such surface is continuous but not necessarily piecewise smooth (which stokes theorem requires). We need to fix this.

We make a surface by continuously deforming a piecewise smooth loop to a point by simply-connected-ness. We take the possibly non-piecewise-smooth surface and consider the minimum distance to the boundary of the open region, call this ε. This is attained by the extreme value theorem since the surface itself is closed. The surface is also compact so if we cover it by open balls of radius ε we can take a finite subcover.

Our original surface is effectively a continuous function $[0,1] \times [0,1] \to D$ where D is our simply connected domain, the first argument says where we are in the deforming the loop process, and the second argument says where we are along the loop itself. Now by uniform continuity, there is some δ such that we can split the domain into squares of length δ and the distance between the corresponding image points will be less than some fixed multiple of ε. Now we pre-suppose that our surface is piecewise linear and not only piecewise smooth, which seems like cheating but it is because if we have path independence over all such paths we can find a function such that our vector field is the grad, proving conservativeness. We now triangulate $[0,1] \times [0,1]$ by things of size δ and then parametrize from these triangles to triangular planar regions. We can then apply stokes theorem on each of these triangles and we are done.

□

We note that divergence is intuitively, if fluid is moving by the vector field, how much it would be coming out of a region, by the divergence theorem.

Similarly by stokes theorem, curl in a sense measures rotation.

If we have smooth scalar fields ϕ, ψ and a volume V with piecewise smooth boundary then by applying the divergence theorem to $\phi\nabla\psi$ we get the identity

$$\int_V \phi\nabla^2\psi + (\nabla\phi) . (\nabla\psi)\, dV = \int_{\partial V} \phi\nabla\psi .ndS$$

Note that this is because by summation convention and the product rule,

$$Div\,(\phi\nabla\psi) = \frac{\partial(\phi\nabla\psi)_i}{\partial x_i} = \phi\frac{\partial(\nabla\psi)_i}{\partial x_i} + (\nabla\psi)_i\frac{\partial\phi}{\partial x_i} = \phi\nabla^2\psi + (\nabla\phi) . (\nabla\psi)$$

Proved similarly, we get Green's second theorem:

$$\int_V \phi\nabla^2\psi - \psi\nabla^2\phi dV = \int_{\partial V} (\phi\nabla\psi - \psi\nabla\phi) .ndS$$

[Lecture 16 ends]

Notational trick: Write if n is a vector, $\frac{\partial f}{\partial n} = \nabla f.n$

Example. If we write f=xyz in a cube with $0 \leq x, y, z \leq 2$ then

$$\int_V \nabla f\ dV = \begin{pmatrix} \int_V \text{yz}\ dV \\ \int_V \text{xz}\ dV \\ \int_V \text{yx}\ dV \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ 8 \end{pmatrix}$$

6 Laplace and poisson equations

For sections 6 and 7 (hehe) they will be introductory. We may see more on tensors in later courses, and more on this in courses on Harmonic analysis.

Poisson's equation says $\nabla^2\phi = f$ where ϕ is an unknown scalar field. Laplace's equation is the special case $f = 0$. We work in a domain D and call f the source term.

A **classical solution** to Poisson's equation or Laplace's equation is one which is 2x continuously differentiable. We will look only for classical solutions. If I say we have the general solution, I mean we have the general classical solution.

Classical solutions are good because if we take the grad of one, it has a continuous div, and its div has continuous components, which means we can use the divergence theorem on it.

Since this is a differential equation we will often impose some contditions. As an example, we might impose that $\phi = g$ on the boundary of D, or that $\frac{\partial\phi}{\partial n} = g$ on the boundary of D.

Example. We will consider laplace's equation in the square $0 \leq x, y \leq 1$ subject to boundary conditions

$$\phi\,(x, 0) = 0,\ \phi\,(x, 1) = 0,\ \phi\,(0, y) = 0,\ \phi\,(1, y) = \sin\,(\pi y)$$

We will find all solutions that can be written in the form $\phi\,(x, y) = p\,(x)\,q\,(y)$. We see that we must have that $q\,(y) = \frac{1}{p(1)}\sin\,(\pi y)$. We know that

$$0 = \nabla^2\phi = \phi_{xx} + \phi_{yy} = pq'' + p''q$$

We now have a differential equation

$$0 = \frac{1}{p\,(1)}\left(p'' - \pi^2 p\right)\sin\,(\pi y)$$

We now know since the other factors are not 0 that we can solve for p.

Lets look for solutions to $\nabla^2\phi = r$ in $\mathbb{R}^3$ with $r \leq 1$. Specifically we will find all solutions depending only on r, where r is the radius. We will impose the condition that $\phi\,(1) = 0$.

Using the formula

$$\nabla^2 f = \frac{1}{h_u h_v h_w}\left\{\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right\}$$

in spherical polar coordinates (where u is r, v and w are θ and ϕ, and the fact that f only depends on r so $f_\theta = f_\phi = 0$ we get

$$\nabla^2\phi = \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial r}\left(r^2 \sin(\theta)\phi'(r)\right) = \phi''(r) + 2\frac{\phi'(r)}{r} = \frac{1}{r}\frac{d^2}{dr^2}\left(r\phi(r)\right)$$

We know have $\frac{1}{r}\frac{d^2}{dr^2}\left(r\phi(r)\right) = r$ so $\frac{d^2}{dr^2}\left(r\phi(r)\right) = r^2$ (we can multiply by r) so $r\phi(r) = \frac{1}{12}r^4 + Ar + B$ and we get $\phi(r) = \frac{1}{12}r^3 + A$. However, we know that the solution must be well defined on our domain and that $\phi(1) = 0$. Therefore the only possible solution is $\phi(r) = \frac{1}{12}\left(r^3 - 1\right)$.

Example. We will find all solutions to $\nabla^2\phi = 0$ in $\mathbb{R}^3$ with $1 \le r \le 2$ where ϕ only depends on r. We will have the condition that $\frac{\partial\phi}{\partial n} = 1$ when $r = 2$ and $\phi = 0$ when $r = 1$. We have $\frac{1}{r}\frac{d^2}{dr^2}\left(r\phi(r)\right) = 0$ which after simplifying gives that $\phi = A + \frac{B}{r}$. This means that $A + B = 0$ by the $r = 1$ condition and we note that at $r = 2$, $n = e_r$ and from how we define $\frac{\partial\phi}{\partial n}$ we know now that $e_r.\nabla\phi = 1$ whenever $r = 2$. We know that in spherical polar coordinates $e_r.\nabla\phi = \frac{\partial\phi}{\partial r}$ from many lectures ago, so we now get the condition $\frac{\partial\phi}{\partial r} = 1$ when $r = 2$. This condition then implies that $B = -4$ and $A = 4$, so we have our solution.

Note that ∇^2 has the property that it is linear, ie $\nabla^2(af + bg) = a\nabla^2 f + b\nabla^2 g$. We sometimes use this to our advantage if, for example, we have an equation $\nabla^2\phi = f$ with $\phi = g$ on the boundary we could find the solutions to

1. $\nabla^2\phi = f$ with $\phi = 0$ on the boundary
2. $\nabla^2\phi = 0$ with $\phi = g$ on the boundary

And add them together.

Assuming the region and its boundary is smooth, we note that by the divergence theorem, in the equation $\nabla^2\phi = f$ with $\frac{\partial\phi}{\partial n} = g$ on the boundary

$$\int_D f\ dV = \int_D div(\nabla\phi)\ dV = \int_{\partial D} \nabla\phi.n\ dS = \int_{\partial D} \text{g}dS$$

So if $\int_D f\ dV = \int_{\partial D} \text{g}dS$ is not satisfied there will be no solutions.

[Lecture 17 ends]

Theorem. Suppose that $\nabla^2\phi = f$ in D and that $\phi = g$ on the boundary of D. Suppose further that the boundary of D is piecewise smooth. Then this has a unique classical solution.

Proof. Let ϕ_1, ϕ_2 both be smooth solutions and $\psi = \phi_1 - \phi_2$. Then by classicalness, $\psi\nabla\psi$ is continuously differentiable. We know that $\nabla^2\psi = 0$ in D and $\psi = 0$ on the boundary. Applying the divergence theorem to this gives that

$$\int_D \nabla.(\psi\nabla\psi)dV = \int_{\partial D}(\psi\nabla\psi).ndS$$

Since ψ is 0 on the boundary these are both equal to 0. We use the identity from a previous lecture

$$\int_V \phi\nabla^2\psi + (\nabla\phi).(\nabla\psi)\,dV = \int_{\partial V}\phi\nabla\psi.ndS$$

To get that

$$\int_D \psi\nabla^2\psi + (\nabla\psi).(\nabla\psi)\,dV = 0$$

But the first term is 0 so

$$\int_D |\nabla\psi|^2 dV = 0$$

By continuity of $\nabla\psi$ (true because we are looking for classical solutions), this means that $\nabla\psi$ is 0 everywhere in D, and so ψ is constant, and this constant is 0 because ψ is 0 on the boundary.

□

Theorem. Suppose that $\nabla^2\phi = f$ in D and that $\frac{\partial\phi}{\partial n} = g$ on the boundary of D. Suppose further that the boundary of D is piecewise smooth. Then this has a unique classical solution (up to differing by a constant).

Proof. Again let ϕ_1, ϕ_2 both be classical solutions and $\psi = \phi_1 - \phi_2$. Then $\nabla\psi.n$ is 0 on the boundary, so in the integral $\int_{\partial V} \psi\nabla\psi.ndS$ which we previously showed is 0, it is again 0 because we get that the integral $\int_{\partial V} \psi 0 dS$ is 0. For the same reason as in the above proof, $\nabla\psi$ is 0 everywhere in D, and so ψ is constant. But this time, the constant does not have to necessarily be 0.

□

We will now consider a method to solve Poisson's equation when everything is radially symmetric (ie, rotationally symmetric, only depending on distance from the origin). The idea is that we guess that we have a solution depending only on r, as we did last lecture, and then by the previous theorems, we know that if this works then we know all the classical solutions.

If $\nabla^2\phi = f(r)$ then define $F(R) \coloneqq \int_{|x|\le R} f(r)\, dV$. We will suppose that $f(r)$ is bounded near $x = 0$ so that we can take a limit to apply the change of variables rule. This is

$$F(R) = \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} f(r)\, r^2 \sin(\theta) d\phi d\theta dr$$

This integral is separable and we get

$$F(R) = 4\pi\int_0^R r^2 f(r)\, dr$$

The problem is we are using ϕ for 2 different things (the angle and the solution). It sucks but it is obvious which one is which.

But also,

$$F(R) = \int_{|x|\le R} \nabla^2\phi dV$$

Since ϕ is a classical solution we can use the divergence theorem to get

$$F(R) = \int_{|x|\le R} \nabla\phi.e_r dS = \int_{|x|=R} \left(\phi'(R)\, e_r\right).e_r dS = \phi'(R)\int_{|x|\le R} dS = 4\pi R^2\phi'(R)$$

Since these are equal to eachother we have $\int_0^R r^2 f(r)\, dr = R^2\phi'(R)$

Therefore, rearranging gives $\phi(r) + c = \int \frac{1}{r^2}\int_0^r t^2 f(t)\, dt dr$

If we impose the standard condition that $\phi(r) \to 0$ as $r \to \infty$ then we can change the indefinite integral to a definite one. We get that

$$\phi(R) = -\int_R^{\infty}\frac{1}{r^2}\int_0^r t^2 f(t)\, dt dr$$

If we have cylindrical symmetry but not spherical symmetry then we can do the same thing. We want to find the integral of f in a cylinder with radius R and height 1.

$$F(R) = 2\pi\int_0^R pf(p)\, dp$$

And also by the divergence theorem

$$F(R) = \int_{\text{The cylinder boundary}} \nabla\phi.e_p dS$$

As the circular faces cancel out so we just integrate over the curved part of the boundary so we dot with e_p. Using the same technique,

$$F(R) = 2\pi R\phi'(R)$$

The result we end up with is

$$\phi'(R) = \frac{1}{R}\int_0^R pf(p)\,dp$$

Now consider a spherically symmetric f that is 0 outside a ball of radius ε. Then by earlier results we know that $\phi'(R) = \frac{F}{4\pi R^2}$ and so for all $R > \varepsilon$ when F is constant we have $\phi(R) = -\frac{F}{4\pi R}$. We will take a limit as $\varepsilon \to 0$ but keeping F fixed (so f gets larger accordingly), we get that $\phi(R) = -\frac{F}{4\pi R}$ if we have a "point source" f at the origin.

[Lecture 18 ends]

Definition. A harmonic function is one that obeys Laplace's equation in some domain V.

Define, assuming that $B_r(a)$ is contained entirely within V, $\overline{\phi}_r(a) = \frac{1}{4\pi r^2}\int_{\partial B_r(a)} \phi(x)\,dS$ where ϕ is harmonic and $dS = V du dv$ as usual.

Take spherical coordinates centered at a. Then

$$\overline{\phi}_r(a) = \frac{1}{4\pi r^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \phi(x)\,r^2 \sin(\theta) d\theta d\phi$$

But r is constant so we just get

$$\overline{\phi}_r(a) = \frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \phi(x)\sin(\theta) d\theta d\phi$$

Since this is a harmonic function, it is continuous. Therefore we can write

$$\frac{\partial}{\partial r}\overline{\phi}_r(a) = \frac{1}{4\pi r^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \nabla\phi(x).e_r r^2 \sin(\theta) d\theta d\phi$$

Now we can use the divergence theorem to turn this into $\nabla^2\phi$ integrated over the inside of the sphere. But this must be 0 by harmonicness. Therefore, $\overline{\phi}_r(a)$ is constant but by continuity goes to 0 as r goes to 0. Therefore $\overline{\phi}_r(a) = \phi(a)$ for all r.

Theorem. (Maximum principle) ϕ cannot achieve a maximum at the center of any open ball and in particular no local maximum, because otherwise we would have that if the open ball has radius r and it is less than its center everywhere on the boundary then we have $\overline{\phi}_r(a) < \phi(a)$ which is a contradiction.

[Lecture 19 ends]

7 Tensors

Now we will define a tensor in two ways (both of which are natural things to talk about) and then show that they are in fact equivalent. After this, we will talk about their properties.

First definition:

A tensor is a map that takes n vectors from a vector space V and returns a real number. The map has the additional property that it is multilinear which means that if we fix all but one of the vectors, such as all but the first one, it is linear, speficially,

$$T(au + bv, w_2, \dots) = aT(u, w_2, \dots) + bT(v, w_2, \dots)$$

For example, the determinant of a matrix is multilinear so this is a tensor.

For our purposes, V is always $\mathbb{R}^n$.

Second definition:

A tensor is something which generalizes vectors and matrices. A vector is a 1D array of numbers and a matrix is a 2D array of numbers. We say a tensor of rank n is an nD array of numbers. We will often write a tensor in terms of its components, for example ε_{ijk} is a 3D matrix which has 1, 0 and -1 in the appropriate positions.

These have the additional property that they transform in the following way when we switch from one orthonormal basis $\{e_i\}$ to another orthonormal basis $\{e_i'\}$. If the matrix R is such that $e_i'.e_j = R_{ij}$, then with the summation convention we will write that

$$T'_{ijk\ldots} = R_{ip}R_{jq}R_{kr}\ldots T_{pqr\ldots}$$

And we will say that the tensor represented with respect to the primed basis always has its components follow this rule, which is exactly the same rule that vectors and matrices both follow.

Theorem. These definitions are equivalent

Proof. First, suppose we have a multilinear map T. Now define

$$T_{i_1\ldots i_n} = T\left(e_{i_1},\ldots,e_{i_n}\right)$$

Now let e' be a new basis so $e_j' = R_{ji}e_i$

Now let T' be such that we define the components as

$$T'_{j_1\ldots j_n} = T\left(e'_{j_1},\ldots,e'_{j_n}\right)$$

Now, by multilinearity,

$$T'_{j_1\ldots j_n} = T\left(R_{j_1i_1}e_{i_1},\ldots,R_{j_ni_n}e_{i_n}\right)$$

Now we can pull out the R's by multilinearity.

$$T'_{j_1\ldots j_n} = R_{j_1i_1}\ldots R_{j_ni_n}T\left(e_{i_1},\ldots,e_{i_n}\right) = R_{j_1i_1}\ldots R_{j_ni_n}T_{i_1\ldots i_n}$$

This is exactly the transformation law in definition 2.

Now suppose definition 2 holds, then we want to show T is a multilinear map.

Suppose T is an object satisfying the transformation law above. Define a map from n vectors to a real number by $T_{i_1\ldots i_n}\left(v_1.e_{i_1}\right)\left(v_2.e_{i_2}\right)\ldots\left(v_n.e_{i_n}\right)$.

We need to show that this is independent of which basis we calculate in, and that this is multilinear.

Suppose we have

$$T'_{j_1\ldots j_n}\left(v_1.e_{j_1}\right)\left(v_2.e_{j_2}\right)\ldots\left(v_n.e_{j_n}\right)$$

Note that

$$v_1.e_{j_1} = v_1.\left(R_{j_1i}\right)e_i = \left(R_{j_1i}\right)\left(v_1.e_i\right)$$

so we have

$$R_{j_1i_1}T'_{j_1\ldots j_n}\left(v_1.e_{i_1}\right)\left(v_2.e_{j_2}\right)\ldots\left(v_n.e_{j_n}\right)$$

since dot products are linear. We can continue in this way for each of the components until we get to

$$R_{j_1i_1}\ldots R_{j_ni_n}{T'}_{i_1\ldots i_n}\left(v_1.e_{i_1}\right)\left(v_2.e_{i_2}\right)\ldots\left(v_n.e_{i_n}\right)$$

But by the transformation law this is just

$$T_{i_1\ldots i_n}\left(v_1.e_{i_1}\right)\left(v_2.e_{i_2}\right)\ldots\left(v_n.e_{i_n}\right)$$

so we have basis independence. Now we just need to show multilinearity.

But now

$$T_{i_1\ldots i_n}\left(v_1.e_{i_1}\right)\left(v_2.e_{i_2}\right)\ldots\left(v_n.e_{i_n}\right)$$

is very clearly linear in each v_i. So it is multilinear. So done.

□

From now on, $\{e_1, e_2, e_3\}$ refers to any right handed orthonormal basis and $\{e'_1, e'_2, e'_3\}$ refers to a different right handed orthonormal basis. Vectors in terms of components are only relative to a specific basis.

As an example, $v = v_j e_j = v'_k e'_k$ implies that $v'_k e'_i . e'_k = v_j e'_i . e_j$ so $v'_k \delta_{ik} = v_j e'_i . e_j$ so $v'_i = v_j e'_i . e_j$ Therefore we have that $v'_i = R_{ij} v_j$ where $R_{ij} = e'_i . e_j$.

Note that R is an orthogonal matrix because

$$RR^T = \begin{pmatrix} {e'_1}^T \\ {e'_2}^T \\ {e'_3}^T \end{pmatrix} \begin{pmatrix} e'_1 & e'_2 & e'_3 \end{pmatrix} = I$$

Wit respect to basis e.

Since R is orthogonal,

$$R_{ik} R_{jk} = \delta_{ij}$$

Since $R^{-1} = R^T$, it follows that

$$v_i = R_{ji} v'_j$$

Note that v and v' are the same vector they just have different components in different frames or bases.

Example. A tensor of rank 1 satisfies $T'_i = R_{ip} T_p$, so tensors of rank 1 are just vectors.

We consider tensors of rank 0 to just be numbers or scalars.

Example. A tensor of rank 2 satisfies $T'_{ij} = R_{ip} R_{jq} T_{pq}$, so $T' = RTR^T$ if T is a matrix. So T and T' are similar matrices.

[Lecture 20 ends]

Example. δ_{ij} is a rank 2 tensor. We define it to be the same for every basis and we want to check that it satisfies the definition of a tensor. We see that

$$R_{ip} R_{jq} \delta_{pq} = R_{ip} R_{jp} = \left(RR^T\right)_{ij} = \delta_{ij} = \delta'_{ij}$$

since it is the same in different bases. So it is a tensor.

Example. ε_{ijk} is a tensor of rank 3. Again we define it to be the same for every basis and check that this makes it still a tensor. We have

$$R_{ip} R_{jq} R_{kr} \varepsilon_{pqr} = \varepsilon_{ijk} \det(R) = \varepsilon_{ijk} = \varepsilon'_{ijk}$$

By determinant properties and the fact that R is rotation. So it is a tensor.

(Non-)Examples:

Consider an object that is given by $(0,0,0)$ with respect to the standard basis and $(1,0,0)$ with respect to some other basis. Clearly, this is not a vector since it cannot change like that with respect to a different basis. So in fact it is not a tensor.

An object that is δ_{ij} in one basis cannot be $-\delta_{ij}$ in another. Intuitively this is because the identity matrix is the identity matrix in every basis.

A component of a vector is not a tensor because it will be different in different bases but rank 0 tensors are the same in every basis.

Note that linear combinations of tensors are tensors, for example lets say $T_{ij} = aA_{ij} + bB_{ij}$. Then we have

$$T'_{ij} = aA'_{ij} + bB'_{ij} = aR_{ip} R_{jq} A_{pq} + bR_{ip} R_{jq} B_{pq} = R_{ip} R_{jq} T_{pq}$$

Recall that A and A' are the same thing with respect to a different basis.

Therefore tensors form a vector space.

The tensor product of two tensors with ranks m and n is a tensor of rank m and n with the rule

$T_{ijk...lmn...} = A_{ijk...}B_{lmn...}$ and T has rank m+n and is written as $A \bigotimes B$.

Definition. We will prove that this is actually a tensor. I will do an example for m=n=2.

$$T'_{ijkl} = A'_{ij} = B'_{kl} = R_{ip}R_{jq}A_{pq}R_{kr}R_{ls}B_{rs} = R_{ip}R_{jq}R_{kr}R_{ls}T_{pqrs}$$

(Contraction) This is about the dodgey summation convention. If we make two of the indices of a tensor be the same then we sum over them so we end up with a tensor of rank 2 lower. For example, T_{ii} is a scalar (rank 0 tensor) equal to the trace of T. It is actually a tensor because we know that trace is basis invariant. And also, if we have something like T_{jii} then we do

$$T'_j = T'_{jii} = R_{jp}R_{iq}R_{ir}T_{pqr} = R_{jp}\delta_{qr}T_{pqr} = R_{jp}T_{pqq} = R_{jp}T_p$$

And the same method works in general.

Definition. The tensor product as above is an outer product. An inner product of two tensors $A_{ijk...}$ and $B_{lmn...}$ of ranks m and n respectively occurs when we multiply them. We do $A_{ijk...}B_{imn...}$ and use the summation convention to get the inner product. This is known as contracting A and B together. This gives a rank m+n-2 tensor.

Example. In the case of 1-tensors this is just a dot product of vectors. In the case of 2-tensors this is just $A^T B$ in matrix notation.

Example. If A is a matrix and v a vector then $A_{ij}v_i$ is another vector.

These are tensors because they are just contractions of the corresponding outer products.

Note that the cross product is a double contraction of 3 tensors: $\varepsilon_{ijk}a_j b_k$.

Suppose that I have an object $T_{ijk...}$ of some rank, then if the outer or inner product of T with any tensor A for some given rank is also a tensor, then in many cases we can prove using indices that T is also a tensor.

[Lecture 21 ends]

As an example, if $T_{ij}v_j$ or $T_{ij}v_k$ is a tensor for all vectors v then T is a tensor.

Proof. Let $w_i = T_{ij}v_j$. Then

$$T'_{ij}v'_j = w'_i = R_{ip}w_p = R_{ip}T_{pq}v_q = R_{ip}T_{pq}R_{jq}v'_j$$

so

$$\left(T'_{ij} - R_{ip}R_{jq}T_{pq}\right)v'_j = 0$$

This is true for EVERY v, so $T'_{ij} = R_{ip}R_{jq}T_{pq}$. Therefore, T is a tensor by definition.

□

As an example, let T be an object of rank 2 and suppose that $T_{ki}A_{jk}$ is a tensor for all tensors A, then T is a tensor. The proof is let $W_{ij} = T_{ki}A_{jk}$, so

$$T'_{ki}A'_{jk} = W'_{ij} = R_{ip}R_{jq}W_{pq} = R_{ip}R_{jq}T_{rp}A_{qr}$$

$$= R_{ip}R_{jq}T_{rp}R_{sq}R_{kr}A'_{sk} = R_{ip}T_{rp}R_{kr}\delta_{js}A'_{sk} = R_{ip}T_{rp}R_{kr}A'_{jk}$$

Therefore

$$\left(T'_{ki} - R_{kr}R_{ip}T_{rp}\right)A'_{jk} = 0$$

for every A', so

$$T'_{ki} - R_{kr}R_{ip}T_{rp} = 0$$

so T is a tensor.

We can generalize this.

Theorem. (Quotient theorem) Suppose we have an object T, and suppose that for all tensors A of a fixed rank, we get a tensor if we take the product TA. Then T is a tensor.

Proof. We will first prove the following special case:

If $A_{k_1 \dots k_r}$ is a set of numbers such that $A_{k_1 \dots k_r} T_{k_1 \dots k_r}$ for every tensor T of rank R, then $A_{k_1 \dots k_r} = 0$

The proof of the special case is as follows. Since T is arbitrary, we will let T be zero everywhere except for one component which we will call $T_{q_1 \dots q_r}$. Now $A_{k_1 \dots k_r} T_{k_1 \dots k_r} = 0$ but every term in the sum vanishes except the $A_{q_1 \dots q_r} T_{q_1 \dots q_r}$ term which means that $A_{q_1 \dots q_r} 1 = 0$ so $A_{q_1 \dots q_r} = 0$. Since T is arbitrary, this holds for every combination of Q's.

Now we will prove the general case.

Suppose $Y_{j_1 \dots j_n} = X_{i_1 \dots i_m} T_{i_1 \dots i_m j_1 \dots j_n}$ for all tensors T in the unprimed frame, and Y is a tensor. Then we want to show that X follows the tensor transformation law.

Now $Y'_{b_1 \dots b_n} = X'_{a_1 \dots a_m} T'_{a_1 \dots a_m b_1 \dots b_n}$. This is subtle – We are defining Y in other coordinate systems this way and then saying that actually Y and T transforms as a tensor.

Write $Y'_{b_1 \dots b_n} = R_{b_1 c_1} \dots R_{b_n c_n} Y_{c_1 \dots c_n}$ and $T'_{a_1 \dots a_m b_1 \dots b_n} = R_{a_1 d_1} \dots R_{a_m d_m} R_{b_1 e_1} \dots R_{b_n e_n} T_{d_1 \dots d_m e_1 \dots e_n}$

Therefore

$$R_{b_1 c_1} \dots R_{b_n c_n} Y_{c_1 \dots c_n} = X'_{a_1 \dots a_m} R_{a_1 d_1} \dots R_{a_m d_m} R_{b_1 e_1} \dots R_{b_n e_n} T_{d_1 \dots d_m e_1 \dots e_n}$$

But by assumption, substitute

$$R_{b_1 c_1} \dots R_{b_n c_n} X_{k_1 \dots k_m} T_{k_1 \dots k_m c_1 \dots c_n} = X'_{a_1 \dots a_m} R_{a_1 d_1} \dots R_{a_m d_m} R_{b_1 e_1} \dots R_{b_n e_n} T_{d_1 \dots d_m e_1 \dots e_n}$$

Now rename all dummy indices.

$$R_{b_1 e_1} \dots R_{b_n e_n} X_{d_1 \dots d_m} T_{d_1 \dots d_m e_1 \dots e_n} = X'_{a_1 \dots a_m} R_{a_1 d_1} \dots R_{a_m d_m} R_{b_1 e_1} \dots R_{b_n e_n} T_{d_1 \dots d_m e_1 \dots e_n}$$

Now multiply both sides by $R_{b_1 f_1} \dots R_{b_n f_n}$ which will result in summing over the b's.

$$R_{b_1 e_1} \dots R_{b_n e_n} X_{d_1 \dots d_m} T_{d_1 \dots d_m e_1 \dots e_n} R_{b_1 f_1} \dots R_{b_n f_n} = X'_{a_1 \dots a_m} R_{a_1 d_1} \dots R_{a_m d_m} R_{b_1 e_1} \dots R_{b_n e_n} T_{d_1 \dots d_m e_1 \dots e_n} R_{b_1 f_1} \dots R_{b_n f_n}$$

But note that $R_{b_i e_i} R_{b_i f_i} = \delta_{e_i f_i}$ so we write

$$\begin{aligned} \delta_{e_1 f_1} \dots \delta_{e_n f_n} X_{d_1 \dots d_m} T_{d_1 \dots d_m e_1 \dots e_n} &= X'_{a_1 \dots a_m} R_{a_1 d_1} \dots R_{a_m d_m} \delta_{e_1 f_1} \dots \delta_{e_n f_n} T_{d_1 \dots d_m e_1 \dots e_n} \\ X_{d_1 \dots d_m} T_{d_1 \dots d_m f_1 \dots f_n} &= X'_{a_1 \dots a_m} R_{a_1 d_1} \dots R_{a_m d_m} T_{d_1 \dots d_m f_1 \dots f_n} \end{aligned}$$

Now write

$$(X_{d_1 \dots d_m} - X'_{a_1 \dots a_m} R_{a_1 d_1} \dots R_{a_m d_m}) T_{d_1 \dots d_m f_1 \dots f_n} = 0$$

And now let the term in the brackets be $Z_{d_1 \dots d_m}$, which we want to show is 0 then we will be done. We have the law

$$Z_{d_1 \dots d_m} T_{d_1 \dots d_m f_1 \dots f_n} = 0$$

For each Z, for any T.

Now if we fix f to specific values, we effectively have T an arbitrary tensor of rank m, and we can use the special case we proved above to conclude.

□

Proposition. If T_{ij} is a tensor then T_{ji} is also a tensor.

Proof.

$$T'_{ji} = R_{jp} R_{iq} T_{pq} = R_{jq} R_{ip} T_{qp}$$

□

Definition. A tensor is symmetric if it is the same when we swap two indices

Definition. A tensor is antisymmetric or alternating if when we interchange indices it multiplies by the sign of the resulting permutation.

Example. The determinant is an alternating tensor – it is a multilinear map from n vectors to a number that flips sign if we interchange two of the indices.

Proposition. A symmetric tensor times an antisymmetric tensor is 0.

Proof. Let S and A be symmetric and antisymmetric respectively. Then $S_{ij}A_{ij} = -S_{ji}A_{ji}$, but these are dummy indices, so this is minus itself and thus 0.

□

Proposition. A rank 2 tensor can be uniquely written as the sum of a symmetric tensor and an antisymmetric tensor.

Proof. Let

$$S_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right), A_{ij} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

But this is unique because

$$T_{ij} + T_{ji} = S_{ij} + A_{ij} + S_{ji} + A_{ji} = 2S_{ij}$$

because S and A are symmetric and antisymmetric respectively.

□

If the tensor has higher rank we can still do this but our uniqueness proof doesn't work anymore.

Example. Suppose we have an antisymmetric tensor A_{ij}. Then define a vector w by

$$w_i = \frac{1}{2}\varepsilon_{ijk}A_{jk}$$

Now

$$\varepsilon_{ijk}w_k = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}A_{lm} = \frac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)A_{lm} = \frac{1}{2}\left(A_{ij} - A_{ji}\right) = A_{ij}$$

Therefore any rank 2 tensor (or matrix) T can be written as $T_{ij} = S_{ij} + \varepsilon_{ijk}w_k$ for some vector W and symmetric tensor X.

[Lecture 22 ends]

Example. Suppose we have C an object of rank 2 and we know that $C_{ij}S_{ij}$ is a rank 0 tensor for any symmetric tensor S. We will show that the symmetric part of C is a tensor.

Proof. Let T be an arbitrary second rank tensor and write $T_{ij} = S_{ij} + A_{ij}$. Let $C^{(S)}$ be the symmetric part of C and $C^{(A)}$ be the antisymmetric part. We will show that $C^{(S)}_{ij}T_{ij}$ is a tensor so we can conclude by the quotient theorem.

$$C^{(S)}_{ij}T_{ij} = C^{(S)}_{ij}\left(S_{ij} + A_{ij}\right) = C^{(S)}_{ij}S_{ij} = \left(C_{ij} - C^{(a)}_{ij}\right)S_{ij} = C_{ij}S_{ij}$$

Which is always a tensor. Since symmetric contracted with antisymmetric gives 0.

□

Lets come up with an example in which C is not a tensor. Consider $C_{ij} = \varepsilon_{1ij}$ in 1 frame and 0 in every other frame. This is clearly not a tensor but it is antisymmetric and $C_{ij}S_{ij}$ is always 0 which is a tensor (antisymmetric thing times symmetric thing).

Definition. An isotropic tensor is one whose components are the same in every basis. Examples are δ_{ij} and ε_{ijk}.

Any tensor of rank 0 is isotropic.

Any tensor of rank 1 is isotropic if and only if it is the 0 vector, because any non-zero vector clearly has different components in different frames.

Proposition. All the isotropic tensors of rank 2 are of the form $\lambda\delta_{ij}$ if we are in $\mathbb{R}^n$ with n>2. If n=2 this is false, as a 90 degree rotation matrix is a counterexample.

Proof. Let T_{ij} be isotropic, ie $R_{ip}R_{jq}T_{pq} = T_{ij}$ for all R in the special orthogonal group of n*n matrices with n>2. Now do the decomposition into symmetric and antisymmetric parts: $T_{ij} = S_{ij} + A_{ij}$. But now note that S and A are isotropic as if we change basis, T does not change, so $T_{ij} = S'_{ij} + A'_{ij}$. But transformations preserve symmetry/antisymmetry:

$$S'_{ji} = R_{jq}R_{ip}S_{qp} = R_{ip}R_{jq}S_{pq} = S'_{ij}$$

and similarly for A, but we know that the T=S+A decomposition is unique, hence S=S', so S is isotropic, and similarly for A.

Now since $n \geq 3$ there is a k component not equal to i or j. So we will rotate by 180 degrees in the (i,k) plane. So our matrix R has elements

$$R_{ii} = -1,\ R_{kk} = -1,\ R_{jj} = 1$$

and all other diagonal entries 1.

The transformation law says that $T_{ij} = R_{ip}R_{jq}T_{pq}$. However, the only term that survives (since R is diagonal) is the one with p=i and q=j, so now (without summation convention), $T_{ij} = R_{ii}R_{jj}T_{ij} = -T_{ij}$. Therefore, since i does not equal j, $T_{ij} = 0$, so T is only non-zero in the diagonal entries.

Now consider i and j and consider a rotation by 90 degrees in the (i, j) plane. So the relevant entries are

$$R_{ii} = 0,\ R_{ij} = 1,\ R_{ji} = -1,\ R_{jj} = 0$$

for all k not i or j $R_{kk} = 1,\ R_{ik} = 0$. Now $T_{ii} = R_{ip}R_{iq}T_{pq}$. But T is diagonal so we can assume p=q. We therefore have being a bit sloppy, I will write things 4 times to mean we are summing over them),

$$T_{ii} = R_{ik}R_{ik}T_{kk} = R_{ii}R_{ii}T_{ii} + R_{ij}R_{ij}T_{jj} + \sum_{k\neq i,j} R_{ik}R_{ik}T_{kk}$$

Now by what the matrix elements are, this is just T_{jj}. So $T_{ii} = T_{jj}$. Therefore T is a multiple of the identity matrix, which is what we set out to prove.

□

Proposition. All the isotropic tensors of rank 3 are of the form $\lambda\epsilon_{ijk}$ if we are in $\mathbb{R}^3$.

Proof. Again, we are assuming $R_{ip}R_{jq}R_{kr}T_{pqr} = T_{ijk}$ for all $R \in SO(3)$. Consider $R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Now consider T_{111}. Then $T_{111} = R_{1p}R_{1q}R_{1r}T_{pqr}$. The only surviving term is the term $p = q = r = 1$, but this implies that $T_{111} = -T_{111}$, and therefore $T_{111} = 0$. The same thing happens for T_{112}, and by a similar argument, to anything with a repeated index. So there are only 6 non-zero components.

Now consider $R = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $T_{123} = R_{1p}R_{2q}R_{3r}T_{pqr}$. The only surviving term in the summation is p=2, q=1 and r=3. So we end up with $T_{123} = -T_{213}$. A similar argument shows antisymmetry in general. Lets verify symmetry, ie $T_{123} = T_{231}$. To do this, apply antisymmetry twice: $T_{123} = -T_{213} = T_{231}$.

□

[Lecture 23 ends]

Example. Lets calculate the integral $I_{ij} = \int_{r<a} r x_i x_j dV$. Note that I_{ij} is a tensor because

$$R_{pi} R_{qj} I_{ij} = \int_{r<a} r R_{pi} R_{qj} \left(x.e_i\right) \left(x.e_j\right) dV = \int_{r<a} r \left(x.e'_p\right) \left(x.e'_q\right) dV = I'_{pq}$$

And it is isotropic because we are integrating over a sphere which is spherically symmetric.

Therefore $I_{ij} = C\delta_{ij}$ for some C.

Now

$$3C = I_{ii} = \int_{r<a} r|x|^2 dV = \int_{r<a} r^3 dV = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^3 r^2 \sin\left(\theta\right) dr d\theta d\phi = 4\pi * \frac{1}{6} a^6$$

Therefore $C = \frac{2}{9}\pi a^6$.

Example. The vector $\int_{r<a} \left(x.b\right) |x| \, x dV$ has i component $b_j \int_{r<a} |x| \, x_j x_i dV$. Which is $\frac{2}{9}\pi a^6 b_j \delta_{ij} = \frac{2}{9}\pi a^6 b_i$, so that is the result of the integral.

Now if x is a vector, then $x_i = R_{ji} x'_j$. Now $\frac{\delta x_i}{\delta x'_k} = \frac{\delta x_i}{\delta x'_j} \frac{\delta x'_j}{\delta x'_k} = R_{ji}\delta_{jk} = R_{ki}$.

Proposition. In fact, if $\circ$ is a placeholder for any tensor expression (which is multilinear and therefore sufficiently smooth for calculus stuff to work), then by the chain rule,

$$\frac{\partial \circ}{\partial x'_i} = \frac{\partial \circ}{\partial x_j} \frac{\partial x_j}{\partial x'_i} = R_{ij} \frac{\partial \circ}{\partial x_j}$$

Now, considered as an operator,

$$\partial'_i = R_{ij} \partial_j$$

Proposition. Div does not depend on basis

Proof.

$$\frac{\partial F'_i}{\partial x'_i} = R_{ij} \frac{\partial}{\partial x_j} \left(R_{ip} F_p\right) = \left(R^T R\right)_{jp} \frac{\partial F_p}{\partial x_j} = \delta_{jp} \frac{\partial F_p}{\partial x_j} = \frac{\partial F_j}{\partial x_j}$$

□

The same thing holds for Curl of course.

[Lecture 24 ends]